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Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. CX-22

A Perturbation Calculation of the Inelastic Scattering of Electrons by Hydrogen Atoms

S. BOROWITZ and M. M. KLEIN

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A PERTURBATION CALCULATION OF THE INELASTIC SCATTERING OF ELECTRONS BY HYDROGEN ATOMS

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Abstract

A perturbation calculation has been made of the inelastic scattering of fast electrons by hydrogen atoms for the 2S and 2P cases, using for the perturbation only the interaction between the incident electron and the bound electron. The results are then compared, to leading order terms in the energy, with calculations by the more customary perturbation scheme. The results show a difference in energy dependence between the two perturbation schemes for both direct and exchange scattering. The present method shows that, in addition, exchange scattering is negligible compared to direct scattering, whereas the conventional perturbation scheme indicates that direct and exchange scattering are of the same order of magnitude. Calculations by the present method are considerably more difficult than those by the conventional method. The results show, however, that exchange scattering may be neglected in the calculations. In addition, the calculations for direct scattering may be simplified considerably with good results, by replacing one of the wave functions in the scattering integral by a wave function which neglects the effect of the nucleus (method of altered states). The calculations by the present method should not, therefore, be much more laborious than those by the conventional method. The present method is probably more accurate than the conventional perturbation scheme, but experimental work will be necessary to determine which method is more valid.

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1. Introduction

The calculation of the scattering of high-speed electrons by hydrogen-like atoms is usually done by the Born approximation[1]. In this method the standard practice has been to take as the perturbation the interaction of the incoming electron with both the bound electron and nucleus[1]. This procedure will be referred to as the method of the asymmetric perturbation since it is not symmetric in the two electrons. Borowitz[2] has suggested that more accurate results may be possible if we take for the perturbation only the interaction between the two electrons. This procedure will be called the method of the symmetric perturbation.

The symmetric perturbation method has been used to calculate the direct and exchange scattered amplitudes for the case of elastic scattering of fast electrons by hydrogen atoms[2] and the results have been compared with calculations by Corinaldesi and Trainor[3] who use the asymmetric perturbation. The comparison indicates that, to the leading order term in the energy, the two methods give the same results for direct scattering, but give different results for exchange scattering.

In order to have a more complete comparison between the two perturbation methods the calculations of Borowitz[2] have been extended to include the case of inelastic scattering of fast electrons by hydrogen atoms. Because of the large amount of mathematical labor the calculations have been limited to the 2S and 2P states. The energy of the incident electron has been taken as large compared to the energy of the hydrogen atom in its ground state but not so large that relativistic effects need be considered. Because of the high energy of the incident electron only leading order terms in the energy have been computed. The results obtained have been compared with those obtained by Corinaldesi and Trainor[3] who use the asymmetric perturbation.

Mathematical preliminaries and the formulation of the scattering integrals to be evaluated will be discussed in Section 2. The scattered amplitude for 2S direct scattering using the symmetric perturbation will be evaluated in Section 3, and compared with the results obtained with the asymmetric perturbation. In Section 4 we consider a simplified method for evaluating the symmetric perturbation integral of Section 3 (method of altered states). Section 5 considers in detail the evaluation of the symmetric perturbation scattering integral of Section 3 for the case of small scattering angles. The calculation of 2S exchange scattering by the symmetric method is taken up in Section 6 and the results compared with those obtained by the asymmetric method. Sections 7 and 8 discuss 2P scattering, direct and exchange, by the symmetric method and a comparison is made with results obtained with the asymmetric perturbation.

2. Mathematical preliminaries

The scattering amplitude for the case of an electron incident upon a hydrogen atom may be obtained from a solution of the Schrödinger equation for this problem, namely

$$(2.1) \qquad \left\{(\Delta_1 + \Delta_2 + 2(E + \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_{12}})\right\} \Psi(r_1, r_2) = 0$$

where

r_1 and r_2 are the distances of the free and bound electrons from the proton;

$r_{12} = |r_1 - r_2|$ is the relative distance between the free electron and the bound electron;

E = total energy of the system, the sum of the energy of the free electron and the hydrogen atom in the ground state.

A system of units has been chosen where $m = \hbar = e = 1$[2]. The unit of length is thus the first Bohr radius, and the unit of energy is twice the ionization energy of the hydrogen atom. The mass of the proton has been taken as infinite so that the wave functions corresponding to the unperturbed Hamiltonian may be written in closed form. Eq. (2.1) represents an incoming wave in electron 1 and has the asymptotic form of an outgoing spherical wave in particles 1 and 2 for r_1 or $r_2 \to \infty$. The direct and exchange scattering amplitudes are then obtained as the coefficients respectively of the asymptotic forms of the outgoing spherical wave for $r_1 \to \infty$ or $r_2 \to \infty$.

The scattering amplitude σ in first order perturbation theory is given by

$$\sigma = -\frac{1}{2\pi}\int \Psi_f^* V \Psi_i \, d\tau_1 \, d\tau_2 \tag{2.2}$$

where

Ψ_i is the wave function for the initial state;

Ψ_f is the wave function for the final state;

V is the perturbation;

$d\tau_1$ and $d\tau_2$ are the volume elements corresponding to the spaces of $\vec{r}_1$ and $\vec{r}_2$.

If the asymmetric perturbation is chosen, i.e.,

$$V = \left(\frac{1}{r_1} - \frac{1}{r_{12}}\right) \tag{2.3}$$

then the initial and final wave functions Ψ_i and Ψ_f are plane waves, whereas if the symmetric perturbation is chosen, i.e.,

$$(2.4) \qquad V = -\frac{1}{r_{12}}$$

then Ψ_i and Ψ_f are Coulomb wave functions[4] (products of plane waves and hypergeometric functions).

We shall denote the scattering amplitude for 2S scattering by unprimed letters, f(direct) and g(exchange). For 2P scattering the corresponding primed letters f' and g' will be used. The subscripts a and s shall be used to denote that the perturbation is asymmetric or symmetric. For the case of 2S scattering the direct scattered amplitude for the asymmetric perturbation is given by

$$(2.5) \qquad f_a = -\frac{\sqrt{2}}{16\pi^2} \int e^{i(\vec{k}_o - \vec{k}_n) \cdot \vec{r}_1} e^{-\frac{3}{2} r_2} (2 - r_2) \left(\frac{1}{r_1} - \frac{1}{r_{12}} \right) d\tau_1 \, d\tau_2 \, .$$

The direct scattered amplitude for the symmetric perturbation is given by

$$(2.6) \qquad f_s = \frac{\sqrt{2}}{16\pi^2} \int e^{i(\vec{k}_o - \vec{k}_n) \cdot \vec{r}_1} e^{-\frac{3}{2} r_2} (2 - r_2) \, {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o \cdot \vec{r}_1)\right) \frac{1}{r_{12}}$$
$$ {}_1F_1\left(-n_2, 1, i(k_n r_1 + \vec{k}_n \cdot \vec{r}_1)\right) d\tau_1 d\tau_2 \, ,$$

where $\vec{k}_o$ and $\vec{k}_n$ are the propagation vectors of the electrons in the initial and final states, and

$$n_1 = 1/ik_o, \quad n_2 = 1/ik_n \, .$$

The ${}_1F_1$ are confluent hypergeometric functions[5]. The magnitude of k_n is obtained, for given k_o, from conservation of energy:

$$(2.7) \qquad k_o^2 - k_n^2 = 3/4 \, .$$

The exchange scattered amplitude for the asymmetric perturbation is

$$(2.8) \qquad g_a = -\frac{\sqrt{2}}{16\pi^2}\int e^{i\vec{k}_o\cdot\vec{r}_1 - i\vec{k}_n\cdot\vec{r}_2}\, e^{-\left(\frac{r_1}{2}+r_2\right)}(2-r_1)\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right) d\tau_1\, d\tau_2 \,.$$

The exchange scattered amplitude for the symmetric perturbation is given by

$$(2.9) \qquad g_s = \frac{\sqrt{2}}{16\pi^2}\int e^{i\vec{k}_o\cdot\vec{r}_1 - i\vec{k}_n\cdot\vec{r}_2}\, e^{-\left(\frac{r_1}{2}+r_2\right)}(2-r_1)\,\frac{1}{r_{12}}\,{}_1F_1\left(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\right)$$

$${}_1F_1\left(-n_2,1,i(k_n r_2+\vec{k}_n\cdot\vec{r}_2)\right) d\tau_1 d\tau_2 \,.$$

For the case of 2P scattering there are three 2P states corresponding to the values 0, ±1 of the magnetic quantum number m. Of these three states only the state corresponding to the value m = 0 need be considered because the other states yield zero scattered amplitude. The direct scattered amplitude for the asymmetric perturbation is given by

$$(2.10) \qquad f'_a = -\frac{\sqrt{2}}{16\pi^2}\int e^{i(\vec{k}_o-\vec{k}_n)\cdot\vec{r}_1}\, e^{-\frac{3}{2}r_2}\, r_2\cos\theta_2\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right) d\tau_1\, d\tau_2 \,.$$

The direct scattered amplitude for the symmetric perturbation is

$$(2.11) \qquad f'_s = \frac{\sqrt{2}}{16\pi^2}\int e^{i(\vec{k}_o-\vec{k}_n)\cdot\vec{r}_1}\, e^{-\frac{3}{2}r_2}\, r_2\cos\theta_2\,\frac{1}{r_{12}}\,{}_1F_1\left(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\right)$$

$${}_1F_1\left(-n_2,1,i(k_n r_1+\vec{k}_n\cdot\vec{r}_1)\right) d\,\tau_1 d\,\tau_2.$$

The exchange scattered amplitude for the asymmetric perturbation is given by

$$(2.12)\quad g_a' = -\frac{\sqrt{2}}{16\pi^2}\int e^{i\vec{k}_o\cdot\vec{r}_1 - i\vec{k}_n\cdot\vec{r}_2}\, e^{-(\frac{r_1}{2}+r_2)}\, r_1\cos\theta_1\left(\frac{1}{r_1}-\frac{1}{r_{12}}\right)d\tau_1\, d\tau_2 .$$

The exchange scattered amplitude for the symmetric perturbation is

$$(2.13)\quad g_s' = \frac{\sqrt{2}}{16\pi^2}\int e^{i\vec{k}_o\cdot\vec{r}_1 - i\vec{k}_n\cdot\vec{r}_2}\, e^{-(\frac{r_1}{2}+r_2)}\, r_1\cos\theta_1\,\frac{1}{r_{12}}$$

$$ {}_1F_1\Big(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\Big)\ {}_1F_1\Big(-n_2,1,i(k_n r_2+\vec{k}_n\cdot\vec{r}_2)\Big)\ d\tau_1\, d\tau_2 .$$

3. Direct 2S scattering

We consider now the evaluation of the integral for f_s, Eq. (2.6). The integration with respect to $d\tau_2$ may be performed by first expanding $\frac{1}{r_{12}}$ in a series of Legendre polynomials[6] ;

$$\frac{1}{r_{12}} = \frac{1}{r_1}\sum_{\ell=0}^{\infty}\left(\frac{r_2}{r_1}\right)^{\ell} P_\ell(\cos\theta_{12}) \qquad r_1 > r_2 ,$$

$$(3.1)$$

$$\frac{1}{r_{12}} = \frac{1}{r_2}\sum_{\ell=0}^{\infty}\left(\frac{r_1}{r_2}\right)^{\ell} P_\ell(\cos\theta_{12}) \qquad r_1 < r_2 ;$$

here

$$(3.2)\quad P_\ell(\cos\theta_{12}) = P_\ell(\cos\theta_1)P_\ell(\cos\theta_2) + 2\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\cos\theta_1)P_\ell^m(\cos\theta_2)\cos m(\phi_1-\phi_2),$$

where θ_{12} is the polar angle between $\vec{r}_1$ and $\vec{r}_2$.

Integration over $d\tau_2$ yields zero except for $m = 0$, $\ell = 0$, by virtue of the orthogonality of the Legendre polynomials. The angular integration for this case yields 4π and Eq. (2.6) becomes

$$(3.3)\quad f_s = \frac{\sqrt{2}}{4\pi}\int e^{i(\vec{k}_o-\vec{k}_n)\cdot\vec{r}_1}\,{}_1F_1\Big(-n_1,1,i(k_o\, r_1-\vec{k}_o\cdot\vec{r}_1)\Big)\;{}_1F_1\Big(-n_2,1,i(k_n r_1+\vec{k}_n\cdot\vec{r}_1)\Big)\,d\tau_1$$

$$\left[\frac{1}{r_1}\int_0^{r_1}(2-r_2)\,e^{-\frac{3}{2}r_2}\,r_2^2\,dr_2+\int_{r_1}^{\infty}r_2(2-r_2)e^{-\frac{3}{2}r_2}dr_2\right].$$

Carrying out the integration with respect to r_2 we obtain

$$(3.4)\qquad f_s = \frac{2\sqrt{2}}{27\pi}\,I_1+\frac{\sqrt{2}}{9\pi}\,I_2$$

where

$$(3.5)\quad I_1 = \int e^{i\vec{q}\cdot\vec{r}_1}\,e^{-\frac{3}{2}r_1}\,{}_1F_1\Big(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\Big)\;{}_1F_1\Big(-n_2,1,i(k_n r_1+\vec{k}_n\cdot\vec{r}_1)\Big)\,d\tau_1$$

$$(3.6)\quad I_2 = \int e^{i\vec{q}\cdot\vec{r}_1}\,e^{-\frac{3}{2}r_1}\,r_1\;{}_1F_1\Big(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\Big)\,{}_1F_1\Big(-n_2,1,i(k_n r_1+\vec{k}_n\cdot\vec{r}_1)\Big)\,d\tau_1$$

and we have set $\vec{q} = \vec{k}_o-\vec{k}_n$. The integrals I_1 and I_2 may be expressed as derivatives of a standard integral I :

$$(3.7)\qquad \left.\begin{aligned} I_1 &= -\frac{\partial}{\partial\lambda}\,I \\ I_2 &= \frac{\partial^2 I}{\partial\lambda^2} \end{aligned}\right|_{\lambda\,=\,3/2}$$

where

$$(3.8)\quad I = \int e^{i\vec{q}\cdot\vec{r}_1}\,e^{-\lambda r_1}\,\frac{1}{r_1}\,{}_1F_1\Big(ia_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\Big)\,{}_1F_1\Big(ia_2,1,i(k_n r_1+\vec{k}_n\cdot\vec{r}_1)\Big)\,d\tau_1\ ,$$

$$a_1 = 1/k_o,\quad a_2 = 1/k_n.$$

The integral I has been evaluated by Nordsieck[7] who obtained

$$(3.9)\qquad I = \frac{2\pi}{\alpha}\, e^{-\pi a_1} \left(\frac{\alpha}{\gamma}\right)^{ia_1} \left(\frac{\gamma+\delta}{\gamma}\right)^{-ia_2} F(1-ia_1, ia_2, 1, x)$$

where

$$(3.10)\qquad \begin{aligned} \alpha &= \frac{1}{2}\,(q^2+\lambda^2) & \beta &= \vec{k}_n \cdot \vec{q} - i\lambda k_n \\ \gamma &= \vec{k}_o \cdot \vec{q} + i\lambda k_o - \alpha & \delta &= k_o k_n + \vec{k}_o \cdot \vec{k}_n - \beta\,, \end{aligned}$$

$$(3.11)\qquad x = \frac{\alpha\delta-\beta\gamma}{\alpha(\gamma+\delta)}\,,$$

and F is the hypergeometric function of its argument. Before carrying out the differentiation of I, the various parameters in Eq. (3.10) are expressed in the form

$$(3.12)\qquad \begin{aligned} \alpha &= \frac{1}{2}\,(k_o^2 + k_n^2 - 2\vec{k}_o \cdot \vec{k}_n + \lambda^2) & \beta &= \vec{k}_o \cdot \vec{k}_n - k_n^2 - i\lambda k_n, \\ \gamma &= \frac{1}{2}\Big((k_o + i\lambda)^2 - k_n^2\Big) & \delta &= k_o\, k_n + k_n^2 + i\lambda k_n. \end{aligned}$$

The quantity x in Eq. (3.11) will generally be of order of magnitude unity in our calculations. It will be convenient, therefore, to define a new variable y :

$$(3.13)\qquad y = 1 - x = \frac{\gamma(\alpha+\beta)}{\alpha(\gamma+\delta)}$$

which will generally be small compared to unity. Insertion of the values of γ and α and of

$$(3.14) \qquad \begin{aligned} (\alpha+\beta) &= \frac{1}{2}\left[k_o^2 - (k_n + i\lambda)^2\right] \\ (\gamma+\delta) &= \frac{1}{2}(k_o + k_n + i\lambda)^2 \end{aligned}$$

yields for y

$$(3.15) \qquad y = \frac{(k_o - k_n)^2 + \lambda^2}{q^2 + \lambda^2} \; .$$

Inspection of Eq. (3.15) shows y is generally of order $\frac{1}{k_o^2}$ except for θ close to 0, where θ is the angle between $\vec{k}_o$ and $\vec{k}_n$. The case $\theta = 0$ will be considered separately.

In order to simplify the differentiation and be able to make estimates of order of magnitude, the hypergeometric function F of the variable x is now expressed in terms of hypergeometric functions of the variable y by [8]

$$(3.16) \quad F(1-ia_1, ia_2, 1, x) = \frac{\Gamma(ia_1 - ia_2)}{\Gamma(ia_1)\Gamma(1-ia_2)} F_1(1-ia_1, ia_2, 1-ia_1+ia_2, y)$$

$$+ \frac{\Gamma(ia_2 - ia_1)}{\Gamma(ia_2)\Gamma(1-ia_1)} y^{i(a_1 - a_2)} F_2(ia_1, 1-ia_2, 1+ia_1-ia_2, y).$$

We shall be interested in the approximate form of Eq. (3.16) for large k_o. Noting that $a_1 \sim 1/k_o$, $a_1 - a_2 \sim \frac{1}{k_o^3}$, $\Gamma(z) \cong \frac{1}{z}$ for small z, and $\Gamma(1+z) = z\,\Gamma(z)$, we obtain

$$(3.17) \qquad F = -\frac{a_1}{b} F_1 + \left(1 + \frac{a_1}{b} - ib\ln y - ia_1 \ln y\right) F_2$$

where $b = a_1 - a_2$. Since the argument y of F_1 and F_2 is small we may expand F_1 and F_2 in the series [9]

$$F_1(1-ia_1,ia_2,1+ib,y) = 1 + \frac{ia_2(1-ia_1)}{1+ib}\,y + \frac{ia_2(1+ia_2)(1-ia_1)(2-ia_1)}{2(1+ib)(2+ib)}\,y^2 + \dots ,$$

$$(3.18)$$

$$F_2(ia_1,1-ia_2,1-ib,y) = 1 + \frac{ia_1(1-ia_2)}{1-ib}\,y + \frac{ia_1(1+ia_1)(1-ia_2)(2-ia_2)}{2(1-ib)(2-ib)}\,y^2 + \dots .$$

Substituting Eq. (3.18) in Eq. (3.17) and retaining only leading order terms we obtain

$$(3.19) \quad F = 1-a_1a_2\,y-ia_2\ln y-a_1a_2\,y\ln y-a_1a_2\,y^2 + \tfrac{1}{2}(a_1a_2)y^2\ln y \quad .$$

Inspection of Eq. (3.19) shows that the leading order terms in $\frac{\partial F}{\partial\lambda}$ and $\frac{\partial^2 F}{\partial\lambda^2}$ come from the term in ln y, so that Eq. (3.19) may be written in the approximate form

$$(3.20) \qquad F = 1-ia_2\ln y \; .$$

We consider now the evaluation of I_1 and I_2, Eq. (3.7), by differentiation of I, Eq. (3.9). Calculation of the derivatives of I shows that the term involving the derivatives of the coefficient of F are small compared to those involving the derivatives of F. We then obtain for I_1 and I_2

$$I_1 = -\frac{\partial I}{\partial\lambda}\bigg|_{\lambda=3/2} = -\frac{c\partial F}{\partial\lambda}\bigg|_{\lambda=3/2} = \frac{4ik_o(1-\cos\theta)c}{\lambda(q^2+\lambda^2)}\bigg|_{\lambda=3/2}$$

$$(3.21)$$

$$I_2 = \frac{\partial^2 I}{\partial\lambda^2}\bigg|_{\lambda=3/2} = c\,\frac{\partial^2 F}{\partial\lambda^2}\bigg|_{\lambda=3/2} = 4ik_o\,\frac{(1-\cos\theta)(q^2+3\lambda^2)c}{\lambda^2(q^2+\lambda^2)^2}\bigg|_{\lambda=3/2} ,$$

where $c = \frac{2\pi}{\alpha} e^{-\pi a_1} (\frac{\alpha}{\gamma})^{ia_1} (\frac{\gamma+\delta}{\gamma})^{-ia_2}$. Substituting Eq. (3.21) into Eq. (3.4) yields for the scattering amplitude f_s.

$$(3.22) \qquad f_s = c \, \frac{32\sqrt{2}\, ik_o(1-\cos\theta)}{81\,\pi} \; \frac{(q^2+9/2)}{(q^2+9/4)^2} \; .$$

For high energies

$$c \cong \frac{2\pi}{\alpha} = \frac{4\pi}{(q^2+9/4)}$$

$$q^2 \cong 2k_o^2(1-\cos\theta) = (2k_o \sin\tfrac{1}{2}\,\theta)^2$$

so that f_s has the limiting form

$$(3.23) \qquad f_s = \frac{16}{81} \; \frac{\sqrt{2}}{k_o^3\,\mu}$$

where $\mu = \sin^2\frac{1}{2}\theta$.

The scattering amplitude for the asymmetric perturbation f_a, Eq. (2.5), has been evaluated by Corinaldesi and Trainor[3] who found

$$(3.24) \qquad f_a = \frac{8\sqrt{2}}{(q^2+9/4)^3} \cong \frac{1}{8} \; \frac{\sqrt{2}}{k_o^6\,\mu^3} \; ,$$

a result decidedly different from Eq. (3.23). For 2S scattering f_s differs from f_a in both energy dependence and angular dependence.

4. Method of altered states

The symmetric perturbation scheme involves considerable mathematical difficulties because the integrals to be evaluated contain the product of two hypergeometric functions. It is desirable, therefore, to consider possible approximations which will simplify the mathematical work and yet yield essentially the same results as the more exact solution. Schwebel[10] considers a one-dimensional scattering problem for which an exact solution may be obtained and shows that the results obtained with the symmetric perturbation method are better than those obtained with the asymmetric perturbation scheme. He then attempts to improve the asymmetric method by taking into account the effect of the center of force (nucleus) upon the initial wave function. This is equivalent, for our case, to replacing the initial plane wave solution by a Coulomb wave function. His results show that the desired improvement is obtained and that the method of the altered initial state gives results in good agreement with those obtained by the symmetric perturbation method.

We shall consider the same type of approximation as employed by Schwebel, but instead of altering the initial wave function for the asymmetric scheme, we shall replace the Coulomb wave function for the final state in the symmetric method by a plane wave function. That this approximation should not change the essential results yielded by the more exact solution may be seen as follows:

The wave function Ψ in Eq. (2.1) is the sum of the unperturbed wave function Ψ_o and the perturbation or scattered wave Ψ_s. The function Ψ_o satisfies the equation

$$(4.1) \qquad H_o \Psi_o = 0$$

where

$$(4.2) \qquad H_o = \Delta_1 + \Delta_2 + 2(E + \frac{1}{r_1} + \frac{1}{r_2}) .$$

Equation (2.1) then becomes

$$(4.3) \qquad H_o \Psi_s = \frac{2}{r_{12}} (\Psi_o + \Psi_s) .$$

If we solve Eq. (4.3), neglecting the perturbation Ψ_s on the right-hand side and using the Green's function appropriate to the operator[11] H_o, we obtain the symmetric scattering integrals of Section 2. Our approximation consists of writing Eq. (4.3) in the form

$$H' \Psi_s = \frac{2}{r_{12}} \Psi_o + 2\left(\frac{1}{r_{12}} - \frac{1}{r_1}\right) \Psi_s , \tag{4.4}$$

where $H' = H_o - \frac{2}{r_1}$. The operator H' is of the simple plane-wave type for coordinate r_1 and has a Green's function G of the form[11]

$$G = \sum_n \frac{1}{4\pi} \phi_n(r_2)\phi_n^*(r_2') \frac{e^{ik_n|r_1 - r_1'|}}{|r_1 - r_1'|} . \tag{4.5}$$

Neglecting the term in Ψ_s on the right-hand side of Eq. (4.4) and solving for the scattering amplitude f_s in the usual way yields

$$f_s = \frac{\sqrt{2}}{16\pi^2} \int e^{i(\vec{k}_o - \vec{k}_n)\cdot\vec{r}_1} e^{-\frac{3}{2} r_2}(2-r_2)\,{}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)\right) \frac{1}{r_{12}}\, d\tau_1\, d\tau_2 . \tag{4.6}$$

Since our approximation has involved the neglect only of a term containing the perturbation Ψ_s, Eq. (4.6) should be of the same order of accuracy as Eq. (2.6). It should be noted that Eq. (4.6) is the same as Eq. (2.6) except that in (4.6) the hypergeometric function for the final state has been replaced by unity.

The integration of Eq. (4.6) with respect to r_2 is done in the same way as for Eq. (2.6) and yields

$$f_s = \frac{2\sqrt{2}}{27\pi} J_1 + \frac{\sqrt{2}}{9\pi} J_2 \tag{4.7}$$

where

$$J_1 = \int e^{i\vec{q}\cdot\vec{r}_1} e^{-\frac{3}{2} r_1}\, {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)\right) d\tau_1 , \tag{4.8}$$

(4.9) $$J_2 = \int e^{i\vec{q}\cdot\vec{r}_1} e^{-\frac{3}{2} r_1} r_1 \, {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)\right) d\tau_1 .$$

The integrals J_1 and J_2 may be evaluated by expressing the function ${}_1F_1$ as a contour integral in the complex plane[12]:

(4.10) $$ {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)\right) = \frac{1}{2\pi i} \oint \left(1+ \frac{1}{v}\right)^{n_1} v^{-1} e^{-i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)v} dv ,$$

where the contour in the v-plane encloses the points 0 and -1. We then have

(4.11) $$J_1 = \frac{1}{2\pi i} \oint \left(1+ \frac{1}{v}\right)^{n_1} v^{-1} dv \int e^{i\vec{K}\cdot\vec{r}_1} e^{-K_o r_1} d\tau_1 ,$$

(4.12) $$J_2 = \frac{1}{2\pi i} \oint \left(1+ \frac{1}{v}\right)^{n_1} v^{-1} dv \int e^{i\vec{K}\cdot\vec{r}_1} e^{-K_o r_1} r_1 \, d\tau_1 ,$$

where

(4.13) $$\vec{K} = \vec{q} + \vec{k}_o v$$

$$K_o = \frac{3}{2} + ik_o v .$$

Eqs. (4.11) and (4.12) may be integrated with respect to $d\tau_1$ by making use of the standard integral[13]

(4.14) $$J = \int e^{i\vec{K}\cdot\vec{r}_1} e^{-K_o r_1} \frac{d\tau_1}{r_1} = \frac{4\pi}{K^2 + K_o^2} .$$

We then have

$$\int e^{i\vec{K}\cdot\vec{r}_1} e^{-K_o r_1} d\tau_1 = \frac{-\partial}{\partial K_o} J = \frac{8\pi K_o}{(K^2+K_o^2)^2}$$

$$\int e^{i\vec{K}\cdot\vec{r}_1} e^{-K_o r_1} r_1 \, d\tau_1 = \frac{\partial^2}{\partial K_o^2} J = \frac{-8\pi(K^2-3K_o^2)}{(K^2+K_o^2)^3} .$$

Equations (4.11) and (4.12) become

$$J_1 = 8\pi \; \frac{1}{2\pi i} \oint \frac{K_o}{(K^2 + K_o^2)^2} \left(1 + \frac{1}{v}\right)^{n_1} v^{-1} \, dv \tag{4.15}$$

$$J_2 = -8\pi \; \frac{1}{2\pi i} \oint \frac{(K^2 - 3K_o^2)}{(K^2 + K_o^2)^2} \left(1 + \frac{1}{v}\right)^{n_1} v^{-1} \, dv \quad . \tag{4.16}$$

The quantity $K^2 + K_o^2$ is, from Eq. (4.12), given by

$$K^2 + K_o^2 = A + Bv , \tag{4.17}$$

where

$$\begin{aligned} A &= q^2 + (3/2)^2 , \\ B &= 2\left(k_o^2 - \vec{k}_o \cdot \vec{k}_n + \tfrac{3}{2}\, i k_o\right) . \end{aligned} \tag{4.18}$$

Substitution of Eqs.(4.15), (4.16) and (4.17) into Eq. (4.7) yields for the scattering amplitude f_s

$$f_s = \frac{8}{9}\sqrt{2}\,\left(\frac{2}{3}\, i k_o \; J_3 + 9 J_4 + 12 k_o \; J_5 - 4 k_o^2 \; J_6\right) , \tag{4.19}$$

where

$$\begin{aligned} J_3 &= \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1}}{(A + Bv)^2} \, dv \\ J_4 &= \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1}}{v(A + Bv)^3} \, dv \\ J_5 &= \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1}}{(A + Bv)^3} \, dv \\ J_6 &= \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1} v}{(A + Bv)^3} \, dv \; . \end{aligned} \tag{4.20}$$

The integrals J_3 through J_6 may be evaluated by replacing the contour around the points 0, -1 by a new contour enclosing the singularity of the denominator $(A+Bv)$. This procedure is justified because the integrals J_3 through J_6 are of order $\frac{1}{v^2}$ or less in the neighborhood of infinity. We may now utilize Cauchy's integral theorem

$$f^{n}(a) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\, dz \tag{4.21}$$

to evaluate our integrals and, noting that a minus sign must be placed in front of the integrals in (4.20) when using the new contour, we obtain

$$\begin{aligned}
J_3 &= \frac{n_1}{A(A-B)} \left(1 - \frac{B}{A}\right)^{n_1} , \\
J_4 &= \frac{1}{2A^3} \left((2+3n_1+n_1^2) - 2n_1(n_1+1)\frac{A}{A-B} + n_1(n_1-1)\frac{A^2}{(A-B)^2} \right) \left(1 - \frac{B}{A}\right)^{n_1} \\
J_5 &= \frac{n_1}{2A^2} \left(\frac{(n_1+1)}{A-B} - \frac{(n_1-1)A}{(A-B)^2} \right) \left(1 - \frac{B}{A}\right)^{n_1} \\
J_6 &= \frac{1}{2A} \frac{n_1(n_1-1)}{(A-B)^2} \left(1 - \frac{B}{A}\right)^{n_1} .
\end{aligned} \tag{4.22}$$

The term $\left(1 - \frac{B}{A}\right)^{n_1}$ occurring in (4.22) approaches unity for large k_o [2] and will therefore be approximated by unity in our work. An examination of the integrals J_3 through J_6 in (4.22) shows that the principal terms in the expression (4.19) for f_s are contributed by J_3 and J_6. Equation (4.19) then yields

$$f_s = \frac{8\sqrt{2}}{9} \left(\frac{2}{3} \frac{ik_o n_1}{A(A-B)} - \frac{2k_o^2 n_1(n_1-1)}{A(A-B)^2} \right) . \tag{4.23}$$

Now we substitute into (4.23) $n_1 = 1/ik_o$ and the values of A and B given by (4.18); then, retaining only leading order terms, we obtain for the limiting form of f_s

$$f_s = \frac{8}{81} \frac{\sqrt{2}\, i}{k_o^3 \mu} \tag{4.24}$$

which is in good agreement with (3.23). This result indicates the validity of the statement by Schwebel that the method of the altered initial state is applicable to physical problems other than the one he considers[14].

5. Small scattering angles

The calculation of the scattering amplitude for small scattering angle θ requires an additional analysis because the orders of magnitude of the various terms involved in the scattering amplitude change. In order to simplify the analysis the calculation will be made explicitly for $\theta = 0$.

For $\theta = 0$, the quantities q^2, y and x have the values

$$q^2 = (k_o - k_n)^2 \tag{5.1}$$

$$y = \frac{(k_o - k_n)^2 + \lambda^2}{(k_o - k_n)^2 + \lambda^2} = 1 \tag{5.2}$$

$$x = 1 - y = 2\,\frac{k_o\, k_n(1 - \cos\theta)}{(k_o - k_n)^2 + \lambda^2} = 0 \;. \tag{5.3}$$

In view of $x = 0$ it is now convenient to calculate the scattering amplitude from (3.9) where the hypergeometric function is expressed in terms of x. If we calculate I_1 and I_2 (see (3.7)) by differentiating I (see (3.9)), an examination of the orders of magnitude of the various terms shows that the principal contribution comes from the terms involving the derivatives of $\frac{2\pi}{\alpha}$. The function F equals 1 for $x = 0$, and the coefficients other then $\frac{2\pi}{\alpha}$ are approximately equal to unity. The scattering amplitude may therefore be obtained from the approximate form of (3.9), namely

$$(5.4) \qquad I = \frac{2\pi}{a} = \frac{4\pi}{q^2 + \lambda^2} \quad .$$

The integrals I_1 and I_2 are then given by

$$(5.5) \qquad I_1 = -\left.\frac{\partial I}{\partial \lambda}\right|_{\lambda=3/2} = \left.\frac{8\pi\lambda}{(q^2 + \lambda^2)^2}\right|_{\lambda=3/2}$$

$$(5.6) \qquad I_2 = \left.\frac{\partial^2 I}{\partial \lambda^2}\right|_{\lambda=3/2} = -\left.\frac{8\pi(q^2 - 3\lambda^2)}{(q^2 + \lambda^2)^3}\right|_{\lambda=3/2} \quad .$$

Substituting the values of I_1 and I_2 into equation (3.4) for the scattering amplitude f_s, we obtain

$$(5.7) \qquad f_s = \left.\frac{32\lambda^2\sqrt{2}}{9(q^2 + \lambda^2)}\right|_{\lambda=3/2} = \frac{8\sqrt{2}}{(q^2 + \frac{9}{4})^3} \cong \frac{512\sqrt{2}}{729} \quad .$$

The value of f_s in (5.7), with q^2 retained, is exactly the same as that given by (3.24) for f_a. This indicates that the direct scattered amplitude for small scattering angles may be correctly evaluated by the use of the asymmetric perturbation. It may be noted mathematically that the approximate form (5.4) of the fundamental integral I is the result obtained by the use of the asymmetric perturbation.

It is of interest to see whether the method of altered states will yield the correct result for $\theta = 0$. An examination of (4.22) shows that the principal contribution to f_s comes from the term in J_4 which does not contain A - B. Neglecting all but this term, we have from (4.19)

$$f_s = 8\sqrt{2}\; J_4 = \frac{8\sqrt{2}}{A^3} = \frac{8\sqrt{2}}{(q^2+\frac{9}{4})^3} \;, \tag{5.8}$$

in agreement with Eq. (5.7).

6a. Exchange 2S scattering

The exchange scattered amplitude g_s for the symmetric perturbation is given by (2.9). In order to evaluate the integral we first express $\frac{1}{r_{12}}$ as a Fourier integral

$$\frac{1}{r_{12}} = \frac{1}{2\pi^2} \int \frac{\exp\left[i\vec{p}\cdot(\vec{r}_1 - \vec{r}_2)\right]}{p^2}\, d\tau_p \;. \tag{6.1}$$

This procedure separates the r_1 and r_2 variables, and (2.9) becomes

$$\begin{aligned} g_s = \frac{\sqrt{2}}{32\pi^4} \int \frac{d\tau_p}{p^2} \int & \exp\left[i(\vec{k}_o + \vec{p})\cdot\vec{r}_1\right] \exp\left[-i(\vec{k}_n + \vec{p})\cdot\vec{r}_2\right] \exp\left[-(r_1/2+r_2)\right] \\ & (2-r_1)\; {}_1F_1\left(-n_1,1,i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)\right) {}_1F_1\left(-n_2,1,i(k_n r_2 + \vec{k}_n\cdot\vec{r}_1)\right) d\tau_1\, d\tau_2 \;. \end{aligned} \tag{6.2}$$

Next, the hypergeometric functions in (6.2) are replaced by contour integrals in the complex plane (see (4.10)):

$${}_1F_1\left(-n_1,1,i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)\right) = \frac{1}{2\pi i} \oint \left(1+\frac{1}{v}\right)^{n_1} \frac{1}{v} \exp\left[-i(k_o r_1 - \vec{k}_o\cdot\vec{r}_1)v\right] dv, \tag{6.3}$$

$${}_1F_1\left(-n_2,1,i(k_n r_2 + \vec{k}_n\cdot\vec{r}_2)\right) = \frac{1}{2\pi i} \oint \left(1+\frac{1}{u}\right)^{n_2} \frac{1}{u} \exp\left[-i(k_n r_2 + \vec{k}_n\cdot\vec{r}_2)u\right] du \;. \tag{6.4}$$

Substitution of (6.3) and (6.4) into (6.2) yields

$$g_s = \frac{\sqrt{2}}{32\pi^4} \int \frac{d\tau_p}{p^2} \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{dv}{v} \frac{1}{2\pi i} \oint \left(1 + \frac{1}{u}\right)^{n_2} \frac{du}{u}$$

$$\int \exp\left[-\left(\tfrac{1}{2} + i\, k_o v\right) r_1\right] \exp\left[i\vec{K}_1 \cdot \vec{r}_1\right] (2 - r_1)\, d\tau_1 \tag{6.5}$$

$$\int \exp\left[-(1 + i\, k_n u) r_2\right] \exp\left[-i\vec{K}_2 \cdot \vec{r}_2\right] d\tau_2 \quad ,$$

where

$$\vec{K}_1 = (1 + v)\, \vec{k}_o + \vec{p}$$

$$\vec{K}_2 = (1 + u)\, \vec{k}_n + \vec{p} \quad . \tag{6.6}$$

The integrals over u and v are also separable and (6.5) may be written in the form

$$g_s = \frac{\sqrt{2}}{32\pi^4} \int \frac{d\tau_p}{p^2} \left(I_v I_{r_1}\right) \left(I_u I_{r_2}\right) \quad , \tag{6.7}$$

where

$$I_v I_{r_1} = \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{dv}{v}\, I_{r_1} \quad ,$$

$$I_u I_{r_2} = \frac{1}{2\pi i} \oint \left(1 + \frac{1}{u}\right)^{n_1} \frac{du}{u}\, I_{r_2} \quad , \tag{6.8}$$

$$I_{r_1} = \int \exp\left[-\left(\tfrac{1}{2} + i k_o v\right) r_1\right] \exp\left[i\vec{K}_1 \cdot \vec{r}_1\right] (2 - r_1)\, d\tau_1 \quad ,$$

$$I_{r_2} = \int \exp\left[-(1 + i k_n u) r_1\right] \exp\left[-i\vec{K}_2 \cdot \vec{r}_2\right] d\tau_2 \quad . \tag{6.9}$$

The integrals I_{r_1} and I_{r_2} may be evaluated by means of the standard integral in (4.14). Using the procedure following (4.14) we obtain

$$\begin{aligned} I_{r_1} &= 16\pi\left[\frac{(1+ik_o v)}{\left(\left(\frac{1}{2}+ik_o v\right)^2+K_1^2\right)^2} - \frac{2\left(\frac{1}{2}+ik_o v\right)^2}{\left(\left(\frac{1}{2}+ik_o v\right)^2+K_1^2\right)^3}\right] \\ I_{r_2} &= \frac{8\pi(1+ik_n u)}{\left((1+ik_n u)^2+K_2^2\right)^2} \ . \end{aligned} \tag{6.10}$$

The denominators in (6.10) may, with the aid of (6.6), be put in the form

$$\begin{aligned} \left(\frac{1}{2}+ik_o v\right)^2+K_1^2 &= A_1+B_1 v \\ (1+ik_n u)^2+K_2^2 &= A_2+B_2 u \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} A_1 &= p^2+k_o^2+2\vec{k}_o\cdot\vec{p}+\frac{1}{4} \ , \\ B_1 &= 2\left(k_o^2+\vec{k}_o\cdot\vec{p}+\frac{i}{2}k_o\right), \end{aligned} \tag{6.12}$$

$$\begin{aligned} A_2 &= p^2+2\vec{k}_n\cdot\vec{p}+k_n^2+1 \\ B_2 &= 2\left(k_n^2+\vec{k}_n\cdot\vec{p}+ik_n\right) \ . \end{aligned} \tag{6.13}$$

Substitution of (6.10) and (6.11) into (6.8) yields

$$\begin{aligned} I_v I_{r_1} &= 16\pi\left(L_1+ik_o L_2-\frac{1}{2}L_3-2ik_o L_4+2k_o^2 L_5\right) \\ I_u I_{r_2} &= 8\pi\left(M_1+ik_n M_2\right) \end{aligned} \tag{6.14}$$

where

$$L_1 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1} \cdot 1/v}{(A_1 + B_1 v)^2}\, dv$$

$$L_2 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1}}{(A_1 + B_1 v)^2}\, dv$$

$$L_3 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1} \cdot 1/v}{(A_1 + B_1 v)^3}\, dv$$

(6.15)

$$L_4 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1}}{(A_1 + B_1 v)^3}\, dv$$

$$L_5 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1} v}{(A_1 + B_1 v)^3}\, dv$$

$$M_1 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{u}\right)^{n_2} \cdot 1/u}{(A_2 + B_2 u)^2}\, du$$

$$M_2 = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{u}\right)^{n_2}}{(A_2 + B_2 u)^2}\, du\ .$$

The integrals in (6.15), except L_1 and M_1, have previously been evaluated (see (4.20) and (4.22). Evaluating L_1 and M_1 by the method explained in Section 4, and utilizing (4.22), we have

$$\begin{gathered}
L_1 = \frac{1}{A_1^2}\left(1 + n_1 - \frac{n_1 A_1}{(A_1 - B_1)}\right) \\
L_2 = \frac{n_1}{A_1}\,\frac{1}{A_1 - B_1} \\
L_3 = \frac{1}{2A_1^3}\left((2+3n_1+n_1^2) - 2n_1(n_1+1)\,\frac{A_1}{A_1 - B_1} + \frac{n_1(n_1-1)A_1^2}{(A_1 - B_1)^2}\right) \\
L_4 = \frac{n_1}{2A_1^2}\left(\frac{n_1+1}{A_1 - B_1} - \frac{(n_1-1)A_1}{(A_1 - B_1)^2}\right) \\
L_5 = \frac{n_1(n_1-1)}{2A_1(A_1 - B_1)^2} \\
M_1 = \frac{1}{A_2^2}\left(1 + n_2 - \frac{n_2 A_2}{A_2 - B_2}\right) \\
M_2 = \frac{n_2}{A_2}\,\frac{1}{A_2 - B_2}\,.
\end{gathered} \tag{6.16}$$

Equation (6.14) then yields

$$\begin{gathered}
I_v I_{r_1} = 4\pi\left[\frac{4(1+n_1)}{A_1^2} - \frac{(2 + 3n_1 + n_1^2)}{A_1^3} + \frac{4(1 - n_1)}{A_1(A_1 - B_1)} - \frac{2(1+n_1)(2-n_1)}{A_1^2(A_1 - B_1)} + \frac{(4ik_o - 8+5n_1 - n_1^2)}{A_1(A_1 - B_1)^2}\right], \\
I_u I_{r_2} = 8\pi\left[\frac{(1 + n_2)}{A_2^2} + \frac{(1 - n_2)}{A_2(A_2 - B_2)}\right].
\end{gathered} \tag{6.17}$$

Substituting the results obtained for $I_v I_{r_1}$ and $I_u I_{r_2}$ into (6.7) for the scattering amplitude, we have

$$g_s = \frac{\sqrt{2}}{\pi^2}\Bigg\{(1+n_2)\Big[4(1+n_1)E_1 - (2+3n_1+n_1^2)E_2 + 4(1-n_1)E_3 - 2(1+n_1)(2-n_1)E_4 + (4ik_o - 8 + 5n_1 - n_1^2)\,E_5\Big] \tag{6.18}$$
$$+(1-n_2)\Big[4(1+n_1)E_6 - (2+3n_1+n_1^2)E_7 + 4(1-n_1)E_8 - 2(1+n_1)(2-n_1)E_9 + (4ik_o - 8 + 5n_1 - n_1^2)\,E_{10}\Big]\Bigg\} ,$$

where

$$E_1 = \int \frac{d\tau_p}{p^2 A_1^2 A_2^2} \qquad E_6 = \int \frac{d\tau_p}{p^2 A_1^2 A_2 (A_2 - B_2)}$$

$$E_2 = \int \frac{d\tau_p}{p^2 A_1^3 A_2^2} \qquad E_7 = \int \frac{d\tau_p}{p^2 A_1^3 A_2 (A_2 - B_2)}$$

$$E_3 = \int \frac{d\tau_p}{p^2 A_1 A_2^2 (A_1 - B_1)} \qquad E_8 = \int \frac{d\tau_p}{p^2 A_1 A_2 (A_1 - B_1)(A_2 - B_2)} \tag{6.19}$$

$$E_4 = \int \frac{d\tau_p}{p^2 A_1^2 A_2^2 (A_1 - B_1)} \qquad E_9 = \int \frac{d\tau_p}{p^2 A_1^2 A_2 (A_1 - B_1)(A_2 - B_2)}$$

$$E_5 = \int \frac{d\tau_p}{p^2 A_1 A_2^2 (A_1 - B_1)^2} \qquad E_{10} = \int \frac{d\tau_p}{p^2 A_1 A_2 (A_1 - B_1)^2 (A_2 - B_2)} .$$

We shall evaluate (6.18) for high values of k_o and will therefore consider only leading order terms. Because of singularities occurring in the integrals in (6.19), their ordersof magnitude with respect to k_o can not be obtained from simple dimensional considerations. Experience gained in working with these integrals shows that the leading order terms are contibuted by the integrals E_3, E_5, E_6 and E_7. Equation (6.18) may then be written

$$g_s = \frac{\sqrt{2}}{\pi^2}\left(4E_3 + 4ik_0\, E_5 + 4\, E_6 - 2E_7\right) \quad . \tag{6.20}$$

The integrals in (6.20) may be evaluated by means of the technique developed by Feynman [15]. In this method a product of terms is expressed as an integral:

$$\frac{1}{A_1 A_2 \ldots A_n} = (n-1)! \int \frac{\delta(x_1 + x_2 + \ldots x_n - 1)\,dx_1\,dx_2 \ldots dx_n}{(A_1 x_1 + A_2 x_2 + \ldots A_n x_n)^n} \qquad x_i \geq 0 \;. \tag{6.21}$$

Powers higher than the first of a given A term are obtained by differentiating both sides of (6.21) with respect to that A term. The integrals in (6.20) may now be written in the form

$$E_3 = 24 \int \frac{x_3\,\delta(x_1 + x_2 + x_3 + x_4 - 1)\,dx_1\,dx_2\,dx_3\,dx_4\,d\tau_p}{(p^2 x_1 + A_1 x_2 + A_2 x_3 + (A_1 - B_1)x_4)^5} \;, \tag{6.22}$$

$$E_5 = 120 \int \frac{x_3 x_4\,\delta(x_1 + x_2 + x_3 + x_4 - 1)\,dx_1\,dx_2\,dx_3\,dx_4\,d\tau_p}{(p^2 x_1 + A_1 x_2 + A_2 x_3 + (A_1 - B_1)x_4)^6} \;, \tag{6.23}$$

$$E_6 = 24 \int \frac{x_2\,\delta(x_1 + x_2 + x_3 + x_4 - 1)\,dx_1\,dx_2\,dx_3\,dx_4\,d\tau_p}{(p^2 x_1 + A_1 x_2 + A_2 x_3 + (A_2 - B_2)x_4)^5} \;, \tag{6.24}$$

$$E_7 = 60 \int \frac{x_2^2\,\delta(x_1 + x_2 + x_3 + x_4 - 1)\,dx_1\,dx_2\,dx_3\,dx_4\,d\tau_p}{(p^2 x_1 + A_1 x_2 + A_2 x_3 + (A_2 - B_2)x_4)^6} \;. \tag{6.25}$$

The method for evaluating integrals like the foregoing is given in the Appendix of Ref. [2]. Inserting the values of A_1, B_1, A_2 and B_2 from (6.12) and (6.13) into Eqs. (6.22)-(6.25) with respect to $d\tau_p$, dx_1, and dx_4, we obtain

$$E_3 = \frac{3\pi^2}{4(k_o + \frac{i}{2})^2}\left[\int \frac{x_3 dx_2 dx_3}{D_o^{5/2}} - \int \frac{x_3 dx_2 dx_3}{C^{5/2}}\right] , \tag{6.26}$$

$$E_5 = \frac{105\pi^2}{8(k_o + \frac{i}{2})^2}\left[\frac{1}{7}\int \frac{x_3(1 - x_2 - x_3) dx_2 dx_3}{D_o^{7/2}} + \frac{2}{35(k_o + \frac{i}{2})^2} \left\{ \int \frac{x_3 dx_2 dx_3}{D_o^{5/2}} - \int \frac{x_3 dx_2 dx_3}{C^{5/2}} \right\}\right] \tag{6.27}$$

$$E_6 = \frac{3\pi^2}{4(k_n + i)^2}\left[\int \frac{x_2 dx_2 dx_3}{D_n^{5/2}} - \int \frac{x_2 dx_2 dx_3}{C^{5/2}}\right] , \tag{6.28}$$

$$E_7 = \frac{15\pi^2}{16(k_n + i)^2}\left[\int \frac{x_2^2 dx_2 dx_3}{D_n^{7/2}} - \int \frac{x_2^2 dx_2 dx_3}{C^{7/2}}\right] , \tag{6.29}$$

where

$$\begin{aligned} C &= (k_o^2 + \tfrac{1}{4})x_2 + (k_n^2 + 1)x_3 - k_o^2 x_2^2 - k_n^2 x_3^2 - 2\vec{k}_o \cdot \vec{k}_n x_2 x_3 , \\ D_o &= \left[(k_o^2 + \tfrac{1}{4}) + (k_o + \tfrac{i}{2})^2\right]x_2 + \left[(k_n^2 + 1) + (k_o + \tfrac{i}{2})^2\right]x_3 - k_o^2 x_2^2 - k_n^2 x_3^2 \\ &\quad - 2\vec{k}_o \cdot \vec{k}_n x_2 x_3 - (k_o + \tfrac{i}{2})^2 \\ D_n &= \left[(k_o^2 + \tfrac{1}{4}) + (k_n + i)^2\right]x_2 + \left[(k_n^2 + 1) + (k_n + i)^2\right]x_3 - k_o^2 x_2^2 - k_n^2 x_3^2 \\ &\quad - 2\vec{k}_o \cdot \vec{k}_n x_2 x_3 - (k_n + i)^2 . \end{aligned} \tag{6.30}$$

In the foregoing equations the integrals containing C in the denominator may be neglected, and in (6.27) the integral containing $D_o^{5/2}$ in the denominator may also be neglected.

The evaluation of the integrals in Eqs. (6.26) - (6.29) is conveniently done by changing to a new set of coordinates u, v given by

$$u = x_2 + x_3 , \qquad v = x_2 - x_3 \quad . \tag{6.31}$$

Then, noting that the transformation of area requires a factor of $\frac{1}{2}$, we write Eqs. (6.26) - (6.29) in the form

$$E_3 = \frac{6\pi^2}{(k_o + \frac{i}{2})^2 q^5} \int_o^1 \int_{-u}^{u} \frac{(u - v)}{(\omega_o - v^2)^{5/2}} \, dudv \tag{6.32}$$

$$E_5 = \frac{60\pi^2}{(k_o + \frac{i}{2})^2 q^7} \int_o^1 \int_{-u}^{u} \frac{(u - v)}{(\omega_o - v^2)^{7/2}} \, dudv \tag{6.33}$$

$$E_6 = \frac{6\pi^2}{(k_n + i)^2 q^5} \int_o^1 \int_{-u}^{u} \frac{(u + v)}{(\omega_n - v^2)^{5/2}} \, dudv \tag{6.34}$$

$$E_7 = \frac{15\pi^2}{(k_n + i)^2 q^7} \int_o^1 \int_{-u}^{u} \frac{(u + v)^2}{(\omega_n - v^2)^{7/2}} \, dudv \tag{6.35}$$

where

$$\begin{aligned} \omega_o &= \frac{1}{q^2} \left[2\left(k_o^2 + k_n^2 + 2(k_o + \tfrac{i}{2})^2 + \tfrac{5}{4}\right) u - (k_n^2 + k_n^2 + 2\vec{k}_o \cdot \vec{k}_n) u^2 - 4(k_o + \tfrac{i}{2})^2 \right] \\ \omega_n &= \frac{1}{q^2} \left[2\left(k_o^2 + k_n^2 + 2(k_n + i)^2 + \tfrac{5}{4}\right) u - (k_o^2 + k_n^2 + 2\vec{k}_o \cdot \vec{k}_n) u^2 - 4(k_n + i)^2 \right] \quad . \end{aligned} \tag{6.36}$$

We have neglected the uv term occurring in the transformation since it is of order $\frac{1}{k_o^2}$ with respect to the other terms.

The integrations involved in (6.32) - (6.35) are elementary but somewhat laborious if exact results are required. Because we are interested only in the principal terms for large k_o some simplifications may be introduced. The integrals have a strong singularity at $u = 1$ for large k_o and, as will be seen, the principal contributions are from those terms containing the highest power of $\omega - u^2$ in the denominator. Carrying out the integration with respect to v and retaining only these terms, we obtain

$$E_3 = \frac{4\pi^2}{(k_o + \frac{i}{2})^2 q^5} \int_o^1 \frac{u^2}{\omega_o(\omega_o - u^2)^{3/2}} \, du \tag{6.37}$$

$$E_5 = \frac{24\pi^2}{(k_o + \frac{i}{2})^2 q^7} \int_o^1 \frac{u^2(1-u)}{\omega_o(\omega_o - u^2)^{5/2}} \, du \tag{6.38}$$

$$E_6 = \frac{4\pi^2}{(k_n + i)^2 q^5} \int_o^1 \frac{u^2}{\omega_n(\omega_n - u^2)^{3/2}} \, du \tag{6.39}$$

$$E_7 = \frac{12\pi^2}{(k_n + i)^2 q^7} \int_o^1 \frac{u^3}{\omega_n(\omega_n - u^2)^{5/2}} \, du \quad . \tag{6.40}$$

The quantities $\omega_o - u^2$ and $\omega_n - u^2$ differ from perfect squares by a small term of order $\frac{1}{k_o^2}$ with respect to the other terms. Making use of the energy relation (2.7), and Eqs. (6.36) for ω_o and ω_n, we obtain

$$\begin{aligned} \omega_o - u^2 &= -4\frac{k_o^2}{q^2}(u - \rho_o)^2 \\ \omega_n - u^2 &= -4\frac{k_n^2}{q^2}(u - \rho_n)^2 \, , \end{aligned} \tag{6.41}$$

where $\rho_o = 1 + \frac{i}{2k_o}$ and $\rho_n = 1 + \frac{i}{k_n}$. Equations (6.37) - (6.40) are now decomposed into partial fractions and integrated term by term. The leading order terms are contributed by those terms having the highest power of $u - \rho_o$ or $u - \rho_n$ in the denominator when evaluated at the upper limit $u = 1$. We thus obtain

(6.42)
$$E_3 = \frac{\pi^2 i}{k_o^3 q^2}$$

$$E_5 = \frac{3}{2} \frac{\pi^2}{k_o^4 q^2}$$

$$E_6 = \frac{1}{4} \frac{\pi^2 i}{k_n^3 q^2}$$

$$E_7 = \frac{3}{32} \frac{\pi^2 i}{k_n^3 q^2} \quad .$$

Substituting the values for the integrals given in (6.42) into Eq. (6.20) for g_s, and noting that k_n differs from k_o by a term of order $\frac{1}{k_o}$, we finally obtain

(6.43)
$$g_s = \frac{173\sqrt{2}\, i}{16\, k_o^3\, q^2} \cong \frac{173\sqrt{2}\, i}{64\, k_o^5\, \mu} \quad .$$

The exchange scattering amplitude g_a for the asymmetric perturbation (see (2.8)), has been evaluated exactly by Corinaldesi and Trainor[3]. The approximate form of their result for large k_o, expressed in our notation, is

(6.44)
$$g_a = -\frac{\sqrt{2}}{2k_o^6} \left(8 - \frac{1}{\mu^2}\right) \quad .$$

This result differs from that for the symmetric case in both the energy dependence (k_o) and the angular dependence (μ).

It is of interest to note that the part of g_a corresponding to $\frac{1}{r_{12}}$ is given by the integrals E_1 and E_2 of Eq. (6.19), since these integrals are the only ones which would occur if plane-wave functions were used for the symmetric method. Direct evaluation of these integrals shows, however, that they are of order $\frac{1}{k_o^6\mu^2}$ (see (6.44)) and are therefore negligible compared to the terms we have evaluated. The asymmetric method apparently fails to yield the correct principal-order term.

If the method of altered states were used for 2S exchange scattering, the Coulomb wave function for the final state in (6.2) would be replaced by a plane wave, and Eq. (6.17) for $I_u I_{r_2}$ would then contain only a term of the form $1/A_2^2$ but no term of the form $\left[A_2(A_2 - B_2)\right]^{-1}$. The method of altered states would therefore yield only some of the leading order terms. Because for exchange scattering the integrals separate naturally, no essential simplifications are gained; the method does reduce the number of integrals to be computed, but the results are less accurate. The method of altered states will not, therefore, be considered further in exchange scattering.

6b. Small scattering angles

Equation (2.9) for the exchange scattered amplitude g_s will now be evaluated for $\Theta = 0$. An examination of the orders of magnitude of the integrals occurring in (6.18) for $\Theta = 0$ shows that now E_1 and E_2 are of order $\frac{1}{k_o^2}$ while integrals previously considered are of order $\frac{1}{k_o^3}$. It would thus appear that, as in direct scattering, the plane-wave terms become dominant for small θ. A further examination of E_1 and E_2 shows, however, that these terms cancel to order $\frac{1}{k_o^2}$ and that the next leading order term is of order $\frac{1}{k_o^4}$. The plane wave terms thus appear to play a negligible role in exchange scattering for both large and small scattering angles.

An examination of the integrals in (6.19) for $\theta = 0$ shows that the four integrals previously considered are leading order terms but that, in addition, the integral E_4 becomes of comparable order of magnitude. The appropriate form of (6.18) for $\theta = 0$ is thus

$$g_s = \frac{\sqrt{2}}{\pi^2} \left(4\, E_3 - 4\, E_4 + 4ik_o\, E_5 + 4\, E_6 - 2\, E_7\right) . \tag{6.45}$$

The evaluation of the integrals in (6.45) for $\theta = 0$ is carried out in exactly the same way as for large θ through the transformation to u, v coordinates. We now note that in Eqs. (6.32) - (6.35), since q^2 is of order $\frac{1}{k_o^2}$ for $\theta = 0$, ω_o and ω_n are of order k_o^4 and much greater than v^2 which is bounded by unity. The term in v^2 in the denominator may therefore be neglected and after carrying through the v integration we obtain

$$E_3 = \frac{12\pi^2}{(k_o + \frac{i}{2})^2} \int_0^1 \frac{u^2}{(\omega_o q^2)^{5/2}}\, du \tag{6.46}$$

$$E_4 = \frac{40\pi^2}{(k_o + \frac{i}{2})^2} \int_0^1 \frac{u^3}{(\omega_o q^2)^{7/2}}\, du \tag{6.47}$$

$$E_5 = \frac{60\pi^2}{(k_o + \frac{i}{2})^2} \int_0^1 \frac{(u^2 - u^3)}{(\omega_o q^2)^{7/2}}\, du \tag{6.48}$$

(6.49) $$E_6 = \frac{12\pi^2}{(k_n + i)^2} \int_0^1 \frac{u^2}{(\omega_n q^2)^{5/2}}\, du$$

(6.50) $$E_7 = \frac{40\pi^2}{(k_n + i)^2} \int_0^1 \frac{u^3}{(\omega_n q^2)^{7/2}}\, du \quad .$$

For $\theta = 0$, the quantities $\omega_o q^2$ and $\omega_n q^2$ may, if we neglect a small term, be expressed as the perfect squares:

(6.51) $$\begin{aligned} \omega_o q^2 &= -4\, k_o^2\, (u - \rho_o)^2 \\ \omega_n q^2 &= -4\, k_n^2\, (u - \rho_n)^2 \quad . \end{aligned}$$

Carrying out the integration with respect to u and noting that, as previously, the principal contribution comes from the upper limit $u = 1$, we obtain

(6.52) $$E_3 = \frac{3}{2}\, \frac{\pi^2 i}{k_o^3} \quad ,$$

(6.53) $$E_4 = \frac{10}{3}\, \frac{\pi^2 i}{k_o^3} \quad ,$$

(6.54) $$E_5 = \frac{5}{2}\, \frac{\pi^2}{k_o^4} \quad ,$$

(6.55) $$E_6 = \frac{3}{32}\, \frac{\pi^2 i}{k_n^3} \quad ,$$

(6.56) $$E_7 = \frac{5\pi^2 i}{96 k_n^3} \quad .$$

Substitution of the foregoing values into Eq. (6.45) for g_s yields

(6.57) $$g_s = \frac{47}{16}\, \sqrt{2}\, \frac{i}{k_o^3} \quad .$$

The result given by Corinaldesi and Trainor for g_a in the limiting case $\theta = 0$ has the form

$$(6.58) \qquad g_a = \frac{40\sqrt{2}}{81} \; \frac{1}{k_o^4} \, .$$

Thus the result given by the asymmetric method is negligible compared to that given by the symmetric method for small as well as for large scattering angles. This is in contrast to direct scattering, where the asymmetric method is valid for small scattering angles.

7a. Direct 2P scattering

We consider now the case of 2P scattering. The direct scattered amplitude for the symmetric perturbation is given by (2.11). Because of the good results obtained with the method of altered states (Section 4) for direct 2S scattering, we shall use this method for the present case. The wave function for the final state is therefore replaced by a plane-wave function, and (2.11) becomes

$$(7.1) \quad f_s' = \frac{\sqrt{2}}{16\pi^2} \int e^{i\vec{q}\cdot\vec{r}_1} e^{-3r_2/2} r_2 \cos\theta_2 \frac{1}{r_{12}} \; {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o \cdot \vec{r}_1)\right) d\tau_1 d\tau_2 .$$

Next we express $\frac{1}{r_{12}}$ as a series of Legendre polynomials (see Section 3) and integrate with respect to $d\tau_2$. All terms vanish except for $m = 0$, $\ell = 1$; noting that the angular integration yields a factor of $\frac{4\pi}{3}$, we obtain

$$f_s' = \frac{\sqrt{2}}{12\pi} \int e^{i\vec{q}\cdot\vec{r}_1} \cos\theta_1 \; {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o \cdot \vec{r}_1)\right) d\tau_1 \left[\frac{1}{r_1^2} \int_0^{r_1} r_2^4 \, e^{-3r_2/2} \, dr_2 + r_1 \int_{r_1}^{\infty} r_2 \, e^{-3r_2/2} \, dr_2 \right] .$$

Carrying out the integration with respect to r_2 yields

$$f_s' = \frac{\sqrt{2}}{9\pi}\left[\frac{64}{27}\int e^{i\vec{q}\cdot\vec{r}_1}\left(1-e^{-3r_1/2}\right){}_1F_1\left(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\right)\frac{\cos\theta}{r_1^2}\,d\tau_1\right.$$

$$(7.3)\qquad -\int e^{i\vec{q}\cdot\vec{r}_1}e^{-3r_1/2}\left(r_1+\frac{8}{3}+\frac{32}{9}\;\frac{1}{r_1}\right)\cos\theta_1$$

$$\left.{}_1F_1\left(-n_1,1,i(k_o r_1-\vec{k}_o\cdot\vec{r}_1)\right)\,d\tau_1\right]\;.$$

We now write the hypergeometric function ${}_1F_1$ as a contour integral in the complex plane (cf. (4.10)) and obtain

$$f_s' = \frac{\sqrt{2}}{9\pi}\left[\frac{64}{27}\;\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\int e^{i\vec{K}\cdot\vec{r}_1}\left(e^{-ik_o v r_1}-e^{-K_o r_1}\right)\frac{\cos\theta_1}{r_1^2}\,d\tau_1\right.$$

$$(7.4)\qquad \left.-\frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\int e^{i\vec{K}\cdot\vec{r}_1}\;e^{-K_o r_1}\left(\frac{32}{9}\,\frac{1}{r_1}+\frac{8}{3}+r_1\right)\cos\theta_1\,d\tau_1\right]$$

where $\vec{K}$ and K_o are defined by (4.13).

The integration with respect to $d\tau_1$ may be carried out directly, but because of the factor $\cos\theta_1$, branch points will occur which will lead to difficulties in carrying out the v integration. The factor $\cos\theta_1$ is therefore eliminated by putting

$$(7.5)\qquad \cos\theta_1\;e^{i\vec{K}\cdot\vec{r}_1} = \frac{\partial}{\partial k}\left(\frac{e^{i\vec{K}'\cdot\vec{r}_1}}{ir_1}\right)\Bigg|_{k=k_o}\;,$$

where $\vec{K}' = \vec{k} + v\vec{k}_o - \vec{k}_n$ and obviously $\vec{K}' = \vec{K}$ for $k = k_o$. The form $\vec{K}'$ has been chosen to avoid a factor of v in the differentiation. Equation (7.4) now takes the form

$$(7.6)\qquad f_s' = \frac{\sqrt{2}}{9\pi i}\left[\frac{64}{27}N_1-\frac{32}{9}N_2-\frac{8}{3}N_3-N_4\right]\;,$$

where

$$(7.7)\qquad N_1 = \frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\frac{\partial}{\partial k}\int e^{i\vec{K}'\cdot\vec{r}_1}\left(e^{-ik_o v r_1} - e^{-K_c r_1}\right)\frac{d\tau_1}{r_1^3}\Bigg|_{k=k_o}$$

$$(7.8)\qquad N_2 = \frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\frac{\partial}{\partial k}\int e^{i\vec{K}'\cdot\vec{r}_1}\,e^{-K_o r_1}\,\frac{d\tau_1}{r_1^2}\Bigg|_{k=k_o}$$

$$(7.9)\qquad N_3 = \frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\frac{\partial}{\partial k}\int e^{i\vec{K}'\cdot\vec{r}_1}\,e^{-K_o r_1}\,\frac{d\tau_1}{r_1}\Bigg|_{k=k_o}$$

$$(7.10)\qquad N_4 = \frac{1}{2\pi i}\oint\left(1+\frac{1}{v}\right)^{n_1}\frac{dv}{v}\frac{\partial}{\partial k}\int e^{i\vec{K}'\cdot\vec{r}_1}\,e^{-K_o r_1}\,d\tau_1\Bigg|_{k=k_o}\ .$$

As indicated, k will be set equal to k_o after differentiation.

We now consider the integration with respect to $d\tau_1$ of the integrals $N_1 \ldots N_4$. We denote the space integrals occurring in $N_1 \ldots N_4$ by the corresponding primed letter. Making use of the standard integral in (4.14) and of the integral

$$(7.11)\qquad \int_0^\infty \left(e^{-ax} - e^{-bx}\right)\frac{dx}{x} = \ln\left(\frac{b}{a}\right)\ ,$$

we obtain

$$N_1' = \int e^{i\vec{K}'\cdot\vec{r}_1}\left(e^{-ik_o v r_1} - e^{-K_o r_1}\right)\frac{d\tau_1}{r_1^3}$$

$$(7.12)\qquad = 2\pi\left[\ln(K_o^2 + K'^2) - \frac{K_o}{iK'}\ln\frac{(K_o - iK')}{(K_o + iK')} - \ln\left((ik_o v)^2 + K'^2\right) + \frac{ik_o v}{iK'}\ln\left(\frac{ik_o v - iK'}{ik_o v + iK'}\right)\right]$$

$$(7.13)\qquad N_2' = \int e^{i\vec{K}'\circ\vec{r}_1}\,e^{-K_o r_1}\,\frac{d\tau_1}{r_1^2} = -\frac{2\pi}{iK'}\ln\left(\frac{K_o - iK'}{K_o + iK'}\right)\ ,$$

$$(7.14) \qquad N_3' = \int e^{i\vec{K}'\cdot\vec{r}_1} \, e^{-K_o r_1} \, \frac{d\tau_1}{r_1} = \frac{4\pi}{K_o^2 + K'^2} ,$$

$$(7.15) \qquad N_4' = \int e^{i\vec{K}'\cdot\vec{r}_1} \, e^{-K_o r_1} \, d\tau_1 = \frac{8\pi K_o}{(K_o^2 + K'^2)^2} .$$

Before carrying out the differentiation with respect to k, several simplifications may be introduced by expressing the logarithmic forms in N_1' and N_2' as definite integrals. Using the integrals

$$(7.16) \qquad \begin{aligned} &\int_x^\infty \frac{dx}{x^2 + a^2} = -\frac{1}{2ia} \ln\left(\frac{x - ia}{x + ia}\right) \\ &\int_x^\infty \frac{x dx}{x^2 + b^2} - \int_x^\infty \frac{x dx}{x^2 + a^2} = \frac{1}{2} \ln(x^2 + a^2) - \frac{1}{2} \ln(x^2 + b^2) \end{aligned}$$

we write N_1' and N_2' as

$$(7.17) \qquad N_1' = 4\pi \left[\int_o^\infty \frac{K_\lambda d\lambda}{K_\lambda^2 + K'^2} - \int_{3/2}^\infty \frac{K_\lambda \, d\lambda}{K_\lambda^2 + K'^2} + K_o \int_{3/2}^\infty \frac{d\lambda}{K_\lambda^2 + K'^2} - ik_o v \int_o^\infty \frac{d\lambda}{K_\lambda^2 + K'^2} \right]$$

$$(7.18) \qquad N_2' = 4\pi \int_{3/2}^\infty \frac{d\lambda}{K_\lambda^2 + K'^2}$$

where $K_\lambda = \lambda + ik_o v$ and λ denotes the variable of integration. Equation (7.17) may be further simplified to

$$(7.19) \qquad N_1' = 4\pi \left[\int_o^{3/2} \frac{\lambda d\lambda}{K_\lambda^2 + K'^2} + \frac{3}{2} \int_{3/2}^\infty \frac{d\lambda}{K_\lambda^2 + K'^2} \right] .$$

A further cancellation occurs when N_1' and N_2' are combined. Hence we introduce this combination N':

$$(7.20)\qquad \begin{aligned} N' &= N_1' - \frac{3}{2} N_2' \\ &= 4\pi \int_0^{3/2} \frac{\lambda d\lambda}{K_\lambda^2 + K'^2} \quad . \end{aligned}$$

We now calculate the derivatives with respect to k of N', N_3' and N_4' and let $k = k_o$ after differentiation. Noting that

$$(7.21)\qquad \frac{\partial}{\partial k}(K'^2)\Big|_{k=k_o} = 2(k_o - k_n \cos\theta + k_o v)$$

we obtain

$$(7.22)\qquad \frac{\partial}{\partial k}(N') = -8\pi(k_o - k_n\cos\theta + k_o v)\int_0^{3/2}\frac{\lambda d\lambda}{(K_\lambda^2 + K^2)^2}$$

$$(7.23)\qquad \frac{\partial}{\partial k}(N_3') = -8\pi\,\frac{(k_o - k_n\cos\theta + k_o v)}{(K_o^2 + K^2)^2}$$

$$(7.24)\qquad \frac{\partial}{\partial k}(N_4') = -32\pi\,\frac{K_o(k_o - k_n\cos\theta + k_o v)}{(K_o^2 + K^2)^3} \quad .$$

The integration with respect to v may now be carried out. The denominator $K_o^2 + K^2$ has the form $A + Bv$, where A and B have been previously defined (Eq. (4.18)); $K_\lambda^2 + K^2$ is of the form $A_\lambda + B_\lambda v$, where A_λ and B_λ are the same as A and B except that λ replaces 3/2. Defining for obvious reasons the combination N:

$$(7.25)\qquad N = N_1 - \frac{3}{2} N_2 \quad ,$$

we obtain instead of (7.7) - (7.10)

$$(7.26) \qquad N = -8\pi k_o \int_0^{3/2} \lambda d\lambda \, \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{1}{v} \, \frac{(\varepsilon + v)}{(A_\lambda + B_\lambda v)^2} \, dv \, ,$$

$$(7.27) \qquad N_3 = -8\pi k_o \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{1}{v} \, \frac{(\varepsilon + v)}{(A + Bv)^2} \, dv \, ,$$

$$(7.28) \qquad N_4 = -32\pi k_o \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{1}{v} \, \frac{(\varepsilon + v)}{(A + Bv)^3} \, dv \, ,$$

where

$$(7.29) \qquad \varepsilon = \frac{1}{k_o} (k_o - k_n \cos \Theta) \quad .$$

Referring to (4.20) and (6.15), we may write the foregoing equations as

$$(7.30) \qquad N = -8\pi k_o \int_0^{3/2} \lambda d\lambda \left[\varepsilon J_3'(\lambda) + J_3(\lambda)\right]$$

$$(7.31) \qquad N_3 = -8\pi k_o (\varepsilon J_3' + J_3)$$

$$(7.32) \qquad N_4 = -32\pi k_o \left[\frac{3}{2} \varepsilon J_4 + \left(\frac{3}{2} + ik_o \varepsilon\right) J_5 + ik_o J_6\right] \quad ,$$

where $J_3 \ldots J_6$ are defined by (4.20) and where

$$(7.33) \qquad J_3' = \frac{1}{2\pi i} \oint \frac{\left(1 + \frac{1}{v}\right)^{n_1}}{(A + Bv)^2} \, \frac{dv}{v} \quad .$$

The notation $J(\lambda)$ indicates that A and B occurring in J are replaced by A_λ and B_λ.

The J integrals are given by (4.22) and the L_1 integral of (6.15). We then obtain

$$(7.34)\quad N = -8\pi k_o \int_o^{3/2} \lambda d\lambda \left[\left(\frac{1+n_1}{A_\lambda^2} - \frac{n_1}{A_\lambda(A_\lambda - B_\lambda)}\right)\varepsilon + \frac{n_1}{A_\lambda(A_\lambda - B_\lambda)}\right],$$

$$(7.35)\quad N_3 = -8\pi k_o \left[\left(\frac{1+n_1}{A^2} - \frac{n_1}{A(A-B)}\right)\varepsilon + \frac{n_1}{A(A-B)}\right]$$

$$(7.36)\quad N_4 = -32\pi k_o \left\{ \frac{3}{2}\varepsilon\left[\frac{(2+3n_1+n_1^2)}{2A^3} - \frac{n_1(n_1+1)}{A^2(A-B)} + \frac{n_1(n_1-1)}{2A(A-B)^2}\right] + \left(\frac{3}{2}+ik_o\varepsilon\right)\left[\frac{n_1(n_1+1)}{2A^2(A-B)} - \frac{n_1(n_1-1)}{2A(A-B)^2}\right] + \frac{ik_o n_1(n_1-1)}{2A(A-B)^2}\right\}.$$

A consideration of the orders of magnitude of the various terms in the foregoing equations shows that, since A is of order k_o^2 and $A-B$ is of order k_o, the leading terms are furnished only by terms of the form $\frac{1}{A(A-B)}$ or $\frac{1}{A(A-B)^2}$. Retaining only principal order terms, Eqs. (7.34) - (7.36) become

$$(7.37)\quad N = -8\pi k_o n_1(1-\varepsilon) \int_o^{3/2} \frac{\lambda d\lambda}{A_\lambda(A_\lambda - B_\lambda)}$$

$$(7.38)\quad N_3 = -\frac{8\pi k_o n_1(1-\varepsilon)}{A(A-B)}$$

$$(7.39)\quad N_4 = 16\pi \frac{k_o(ik_o n_1)(1-\varepsilon)}{A(A-B)^2}.$$

Equation (7.37) may be integrated with respect to λ most simply by noting that the limit $\frac{3}{2}$ is small compared to k_o and that

$$A_\lambda = q^2 + \lambda^2 \cong q^2$$

$$A_\lambda - B_\lambda = k_n^2 - k_o^2 + \lambda^2 - 2i\lambda k_o \cong -2i\lambda k_o$$

$$(7.40)\qquad \begin{aligned} N &= -8\pi k_o n_1(1-\varepsilon)\int_0^{3/2}\frac{\lambda d\lambda}{q^2(-2ik_o\lambda)} \\ &= -\frac{6\pi(ik_o n_1)(1-\varepsilon)}{q^2 k_o} \;. \end{aligned}$$

Substituting the values of A and A - B into Eqs. (7.38) and (7.39), and noting that $n_1 = 1/ik_o$, we finally obtain

$$(7.41)\qquad N = -\frac{6\pi(1-\varepsilon)}{q^2 k_o}$$

$$(7.42)\qquad N_3 = -\frac{8}{3}\,\frac{\pi(1-\varepsilon)}{q^2 k_o}$$

$$(7.43)\qquad N_4 = -\frac{16}{9}\,\frac{\pi(1-\varepsilon)}{q^2 k_o} \;.$$

Insertion of these values into Eq. (7.6) for the scattering amplitude yields

$$(7.44)\qquad f_s' = \frac{16}{27}\,\frac{\sqrt{2}\,i}{q^2\,k_o}\,(1-\varepsilon) \;.$$

Since

$$1-\varepsilon = \frac{k_n}{k_o}\cos\theta \cong \cos\theta$$

$$q^2 \cong 2k_o^2(1-\cos\theta) = 4k_o^2\mu \;,$$

(7.44) may be written as

$$(7.45)\qquad f_s' = \frac{8\sqrt{2}\,i}{27k_o^3}\,\frac{\cos\theta}{(1-\cos\theta)} = \frac{4\sqrt{2}\,i}{27k_o^3}\left(\frac{1-2\mu}{\mu}\right) \;.$$

The scattered amplitude f_a' (cf. (2.10)) has been worked out exactly by Corinaldesi and Trainor[3]; their result is

(7.46) $$f_a' = \frac{12\sqrt{2}\,i(k_o - k_n\cos\theta)}{q^2(q^2+\frac{9}{4})^3} \cong \frac{3\sqrt{2}}{32}\;\frac{i}{k_o^7\mu^3} \; .$$

Equation (7.46) differs considerably from our results; the energy variation is of order $\frac{1}{k_o^7}$ whereas our result gives $\frac{1}{k_o^3}$. In addition, the angular variation for the asymmetric perturbation is opposite to that given by the symmetric perturbation.

7b. Small scattering angles

The scattering amplitude f_s' for $\theta = 0$ can be evaluated very simply by a consideration of the orders of magnitude in Eqs. (7.34) - (7.36). For $\theta = 0$, the principal contribution comes from those terms which contain only A terms and not A - B terms in the denominator. Equations (7.34) - (7.36) are then

(7.47) $$N = -8\pi k_o\varepsilon \int_o^{3/2} \frac{\lambda d\lambda}{A_\lambda^2}$$

(7.48) $$N_3 = -\frac{8\pi k_o\varepsilon}{A^2}$$

(7.49) $$N_4 = -\frac{48\pi k_o\varepsilon}{A^3} \; .$$

Integrating Eq. (7.47) we obtain

(7.50) $$\begin{aligned} N &= -8\pi k_o\varepsilon \int_o^{3/2} \frac{\lambda d\lambda}{(q^2+\lambda^2)^2} \\ &= -\frac{9\pi k_o\varepsilon}{(q^2+\frac{9}{4})q^2} \end{aligned}$$

Substitution of Eqs. (7.48), (7.49) and (7.50) into Eq. (7.6) for the scattering amplitude yields

$$
\begin{aligned}
f_s' &= \frac{12\sqrt{2}\, i k_o \varepsilon}{q^2\left(q^2+\frac{9}{4}\right)^3} \\
&= \frac{12\sqrt{2}\, i (k_o - k_n \cos\theta)}{q^2\left(q^2+\frac{9}{4}\right)^3} \cong \frac{1536}{729}\, i k_o \,.
\end{aligned}
\tag{7.51}
$$

Equation (7.51) is in exact agreement with Eq. (7.46) for the asymmetric perturbation and indicates that, for 2P direct scattering as well as for 2S direct scattering, the plane wave terms are dominant for small scattering angles. The indications are therefore that valid results may be obtained at small scattering angles by the asymmetric method for all inelastic direct scattering.

8a. Exchange 2P scattering

The exchange scattered amplitude for the 2P case using the symmetric perturbation is given by Eq. (2.13). The procedure for evaluating the integral is very similar to that used in 2S exchange scattering. We first express $\frac{1}{r_{12}}$ as a Fourier integral and (2.13) becomes

$$
\begin{aligned}
g_s' = \frac{\sqrt{2}}{32\pi^4} \int \frac{d\tau_p}{p^2} \int & e^{i(\vec{k}_o + \vec{p})\cdot\vec{r}_1}\, e^{-i(\vec{k}_n + \vec{p})\cdot\vec{r}_2}\, e^{-(r_1/2 + r_2)}\, r_1 \cos\theta_1 \\
& {}_1F_1\left(-n_1, 1, i(k_o r_1 - \vec{k}_o \cdot \vec{r})\right)\, {}_1F_1\left(-n_2, 1, i(k_n r_2 + \vec{k}_n \cdot \vec{r}_2)\right) d\tau_1\, d\tau_2 \,.
\end{aligned}
\tag{8.1}
$$

The hypergeometric functions in (8.1) are now expressed as contour integrals (Eqs. (6.3) and (6.4)), and (8.1) becomes

$$g_s' = \frac{\sqrt{2}}{32\pi^4} \int \frac{d\tau_p}{p^2} \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{dv}{v} \frac{1}{2\pi i} \oint \left(1 + \frac{1}{u}\right)^{n_2} \frac{du}{u}$$

(8.2)

$$\int e^{-(\frac{1}{2} + ik_o v)r_1} e^{i\vec{K}_1 \cdot \vec{r}_1} r_1 \cos\theta \, d\tau_1 \int e^{-(1+ik_n u)r_2} e^{-i\vec{K}_2 \vec{r}_2} d\tau_2 \, ,$$

where $\vec{K}_1$ and $\vec{K}_2$ are defined by (6.6). Because of the separation of the integrals over u and v and the integrals over $d\tau_1$ and $d\tau_2$, Eq. (8.2) may be written as

$$g_s' = \frac{\sqrt{2}}{32\pi^4} \int \frac{d\tau_p}{p^2} (I_v I_{r_1}') (I_u I_{r_2}) \tag{8.3}$$

where

$$I_v I_{r_1}' = \frac{1}{2\pi i} \oint \left(1 + \frac{1}{v}\right)^{n_1} \frac{dv}{v} I_{r_1}'$$

(8.4)

$$I_u I_{r_2} = \frac{1}{2\pi i} \oint \left(1 + \frac{1}{u}\right)^{n_2} \frac{du}{u} I_{r_2}$$

$$I_{r_2}' = \int e^{-(\frac{1}{2} + ik_o v)r_1} e^{i\vec{K}_1 \cdot \vec{r}_1} r_1 \cos\theta_1 \, d\tau_1$$

(8.5)

$$I_{r_2} = \int e^{-(1 + ik_n u)r_2} e^{-i\vec{K}_2 \cdot \vec{r}_2} d\tau_2 \, .$$

The integral I_{r_2} has been previously evaluated (see (6.10). To evaluate I_{r_1}' we first remove the $\cos\theta_1$ factor by the method explained in Section 7 (see (7.5)). The integral I_{r_1}' then becomes

$$I_{r_1}' = \frac{1}{i} \frac{\partial}{\partial k} \int e^{-(\frac{1}{2} + ik_o v)r_1} e^{i\vec{K}_1' \cdot \vec{r}_1} d\tau_1 \Big|_{k=k_o} \, , \tag{8.6}$$

where $\vec{K}_1' = \vec{K} + v\vec{k}_o + \vec{p}$ and obviously $\vec{K}_1' = \vec{K}_1$ for $k = k_o$. The integral in (8.6) is now in standard form (see (4.14)), and we obtain

$$I_{r_1}' = \frac{1}{i}\frac{\partial}{\partial k}\left[\frac{8\pi(\frac{1}{2} + ik_o v)}{\left((\frac{1}{2} + ik_o v)^2 + K_1'^2\right)^2}\right]\Bigg|_{k=k_o} . \tag{8.7}$$

Carrying out the differentiation with respect to k and noting that

$$\frac{\partial}{\partial k}(K_1'^2)\Big|_{k=k_o} = \frac{2}{k_o}\left[k_o^2\,(1+v) + \vec{k}_o \cdot \vec{p}\right] \tag{8.8}$$

we have from (8.7)

$$I_{r_1}' = \frac{32\pi i(\frac{1}{2} + ik_o v)\left[k_o^2(1+v) + \vec{k}_o \cdot \vec{p}\right]}{k_o\left[(\frac{1}{2} + ik_o v)^2 + K_1^2\right]^3} . \tag{8.9}$$

For convenience we write down the integral for I_{r_2}:

$$I_{r_2} = \frac{8\pi(1 + ik_n u)}{\left[(1 + ik_n u)^2 + K_2^2\right]^2} . \tag{6.10}$$

The foregoing equations for I_{r_1}' and I_{r_2} have a form similar to that of (6.10), and when substituted in (8.4) yield

$$\begin{aligned} I_v I_{r_1}' &= \frac{32\pi i}{k_o}\left[\frac{1}{2}(k_o^2 + \vec{k}_o \cdot \vec{p})L_3 + ik_o\left((k_o^2 + \vec{k}_o \cdot \vec{p}) - \frac{ik_o}{2}\right)L_4 + ik_o^3\,L_5\right] \\ I_u I_{r_2} &= 8\pi(M_1 + ik_n\,M_2) \ , \end{aligned} \tag{8.10}$$

where the L and M integrals are defined in (6.15). If their values, given by (6.16), are put into (8.10) we obtain, after some algebra and rearrangement of terms,

$$I_v I'_{r_1} = \frac{4\pi i}{k_o}\left[\frac{2(1+n_1)(2+n_1)(k_o^2+\vec{k}_o\cdot\vec{p})}{A_1^3} - \frac{2(1-n_1^2)}{A_1^2} + \frac{3n_1(1-n_1)}{A_1(A_1-B_1)} + \frac{2(1+n_1)(1-2ik_o)}{A_1^2(A_1-B_1)}\right.$$

$$\left. + \frac{(1-n_1)(2-n_1)}{(A_1-B_1)^2} - \frac{(1-n_1)(4k_o^2+4ik_o-1)}{A_1(A_1-B_1)^2}\right] \quad ; \tag{8.11}$$

$$I_u I_{r_2} = 8\pi\left(\frac{(1+n_2)}{A_2^2} + \frac{(1-n_2)}{A_2(A_2-B_2)}\right) \quad .$$

Substitution of the results obtained for $I_v I'_{r_1}$ and $I_u I_{r_2}$ in Eq. (8.3) for the scattering amplitude yields

$$g'_s = \frac{\sqrt{2}}{\pi^2}\,\frac{i}{k_o}\left\{(1+n_2)\left[(1+n_1)(2+n_1)F_1 - 2(1-n_1^2)F_2 + 3n_1(1-n_1)F_3 + 2(1+n_1)(1-2ik_o)F_4\right.\right.$$

$$\left. + (1-n_1)(2-n_1)F_5 - (1-n_1)(4k_o^2+4ik_o-1)F_6\right]$$

$$+ (1-n_2)\left[(1+n_1(2+n_1)F_7 - 2(1-n_1^2)F_8 + 3n_1(1-n_1)F_9\right. \tag{8.12}$$

$$+ 2(1+n_1)(1-2ik_o)F_{10} + (1-n_1)(2-n_1)F_{11}$$

$$\left.\left. - (1-n_1)(4k_o^2+4ik_o-1)F_{12}\right]\right\} \quad ,$$

where

$$
\begin{array}{llll}
F_1 = \int \frac{2(k_o^2 + \vec{k}_o \cdot \vec{p})d\tau_p}{p^2 A_1^3 A_2^2} & & F_7 = \int \frac{2(k_o^2 + \vec{k}_o \cdot \vec{p})d\tau_p}{p^2 A_1^3 A_2 (A_2 - B_2)} \\
F_2 = \int \frac{d\tau_p}{p^2 A_1^2 A_2^2} & & F_8 = \int \frac{d\tau_p}{p^2 A_1^2 A_2 (A_2 - B_2)} \\
F_3 = \int \frac{d\tau_p}{p^2 A_1 A_2^2 (A_1 - B_1)} & & F_9 = \int \frac{d\tau_p}{p^2 A_1 A_2 (A_1 - B_1)(A_2 - B_2)} \\
F_4 = \int \frac{d\tau_p}{p^2 A_1^2 A_2^2 (A_1 - B_1)} & & F_{10} = \int \frac{d\tau_p}{p^2 A_1^2 A_2 (A_1 - B_1 (A_2 - B_2)} \\
F_5 = \int \frac{d\tau_p}{p^2 A_2^2 (A_1 - B_1)^2} & & F_{11} = \int \frac{d\tau_p}{p^2 A_2 (A_1 - B_1)^2 (A_2 - B_2)} \\
F_6 = \int \frac{d\tau_p}{p^2 A_1 A_2^2 (A_1 - B_1)^2} & & F_{12} = \int \frac{d\tau_p}{p^2 A_1 A_2 (A_1 - B_1)^2 (A_2 - B_2)} \; .
\end{array}
\tag{8.13}
$$

An examination of the integrals in (8.13) shows that the leading order terms in (8.12) are contributed by the integrals F_6 and F_7. Thus for large k_o, (8.12) is approximately

$$
g_s' = \frac{\sqrt{2}}{\pi^2} \frac{i}{k_o} (-4k_o^2 F_6 + 2 F_7) \; . \tag{8.14}
$$

The integral F_6 is the same as the integral E_5 which has been evaluated (cf. (6.42)). We therefore have

$$F_6 = \frac{3}{2} \frac{\pi^2}{k_o^4 q^2} \quad . \tag{8.15}$$

To evaluate F_7 we first eliminate the factor $\vec{k}_o \cdot \vec{p}$ from the numerator by putting

$$\frac{2(k_o^2 + \vec{k}_o \cdot \vec{p})}{A_1^3} = - \frac{k_o}{2} \frac{\partial}{\partial k_o} \left(\frac{1}{A_1^2} \right) \quad . \tag{8.16}$$

The integral F_7 then becomes

$$F_7 = - \frac{k_o}{2} \frac{\partial}{\partial k_o} \int \frac{d\tau_p}{p^2 A_1^2 A_2 (A_2 - B_2)} \quad . \tag{8.17}$$

Using the Feynman technique, we obtain

$$F_7 = - 12 k_o \frac{\partial}{\partial k_o} \int \frac{x_2 \delta(x_1 + x_2 + x_3 + x_4 - 1) dx_1 dx_2 dx_3 dx_4 d\tau_p}{\left[p^2 x_1 + A_1 x_2 + A_2 x_3 + (A_2 - B_2) x_4\right]^5} \quad . \tag{8.18}$$

Integrating with respect to dx_1 and $d\tau_p$ we obtain

$$F_7 = - \frac{15\pi^2 k_o}{16} \frac{\partial}{\partial k_o} \int \frac{x_2\ dx_2\ dx_3\ dx_4}{\left[k_o^2 + \frac{1}{4}) x_2 + (k_n^2 + 1) x_3 - k_o^2 x_2^2 - k_n^2 x_3^2 - 2\vec{k}_o \cdot \vec{k}_n x_2 x_3 - (k_n + i)^2 x_4\right]^{\frac{7}{2}}} \tag{8.18a}$$

Differentiation with respect to k_o and integration with respect to x_4 yields

$$F_7 = \frac{15\pi^2}{8(k_n + i)^2} \left[\int \frac{x_2^2 (k_o^2 - k_o^2 x_2 - \vec{k}_o \cdot \vec{k}_n x_3) dx_2 dx_3}{D_n^{7/2}} - \int \frac{x_2^2 (k_o^2 - k_o^2 x_2 - \vec{k}_o \cdot \vec{k}_n x_3) dx_2 dx_3}{C^{7/2}} \right] \tag{8.19}$$

where C and D_n are defined by (6.30). In (8.19) the integral containing $C^{7/2}$ in the denominator is negligible compared to the other integral.

We now transform to u, v coordinates (see (6.31)) and obtain, noting that odd powers of v disappear upon integration,

$$(8.20)\quad F_7 = \frac{15\pi^2}{(k_n+i)^2 q^7}\left\{ k_o^2 \int_0^1 \int_{-u}^{u} \frac{\left[2(u^2+v^2)-(u^3+3uv^2)\right]}{(\omega_n - v^2)^{7/2}}\, dudv - \vec{k}_o \cdot \vec{k}_n \int_0^1\int_{-u}^{u} \frac{(u^3-uv^2)}{(\omega_n - v^2)^{7/2}}\, dudv \right\}$$

where ω_n is given by (6.36). Integrating with respect to v and retaining only the highest power of $\omega_n - u^2$ in the denominator, we have

$$(8.21)\quad F_7 = 24\, \frac{\pi^2 k_o^2}{(k_n+i)^2 q^7} \int_0^1 \frac{u^3(1-u)}{\omega_n(\omega_n - u^2)^{5/2}}\, du \ .$$

Expressing $\omega_n - u^2$ as a perfect square, integrating with respect to u, and retaining only leading order terms, we obtain

$$(8.22)\qquad F_7 = \frac{3}{16}\, \pi^2\, \frac{k_o^2}{k_n^4 q^2} \ .$$

Now we substitute the values of F_6 and F_7 into Eq. (8.14) for the scattering amplitude. Then

$$(8.23)\qquad g_s' = -\frac{45}{8}\, \frac{\sqrt{2}\, i}{k_o^3\, q^2} \cong -\frac{45}{32}\, \frac{\sqrt{2}\, i}{k_o^5\, \mu} \ .$$

The exchange scattering amplitude for the asymmetric perturbation (2.12) has been evaluated by Corinaldesi and Trainor[3]. The approximate form of their results, for large k_o, is

$$(8.24)\qquad g_a' = -\frac{\sqrt{2}\, i}{8k_o^7}\left(32 - \frac{5}{\mu^2}\right) \ .$$

This result differs from that for the symmetric case in both energy dependence and angle dependence.

8b. Small scattering angles

An examination of (8.12) for $\theta = 0$ shows that F_6 and F_7 remain leading order terms but that, in addition, the integrals F_1, F_2 and F_4 become of comparable order of magnitude. The appropriate form of (8.12) for $\theta = 0$ is then

$$g_s' = \frac{\sqrt{2}}{\pi^2}\,\frac{i}{k_o}\,(2F_1 - 2F_2 - 4ik_oF_4 - 4k_o^2F_6 + 2F_7) \quad . \tag{8.25}$$

The integrals F_4 and F_6 are the same as E_4 and E_5 which have been evaluated for $\theta = 0$ (see (6.53) and (6.54)); they are

$$F_4 = \frac{10}{3}\,\frac{\pi^2 i}{k_o^3} \tag{8.26}$$

$$F_6 = \frac{5}{2}\,\frac{\pi^2}{k_o^4} \quad . \tag{8.27}$$

The integral F_7 may be evaluated for $\theta = 0$ by use of (8.20). Neglecting the v^2 term in the denominator and noting that for $\theta = 0$, $\vec{k}_o \cdot \vec{k}_n$ may be approximated by $k_o^2 - \frac{3}{8}$, we have

$$F_7 = \frac{30\pi^2}{(k_n + i)^2}\left\{k_o^2\int_o^1\int_{-u}^{u}\frac{(1-u)(u^2+v^2)}{(\omega_n\,q^2)^{7/2}}\,dudv + \frac{3}{8}\int_o^1\int_{-u}^{u}\frac{u(u^2-v^2)}{(\omega_n q^2)^{7/2}}\,dudv\right\} . \tag{8.28}$$

Here the second integral on the right-hand side is negligible compared with the first integral. Integrating with respect to v we obtain

$$F_7 = \frac{80\pi^2}{(k_n + i)^2}\,k_o^2\int_o^1\frac{(u^3 - u^4)}{(\omega_n q^2)^{7/2}}\,du \tag{8.29}$$

where $\omega_n q^2$ is given by (6.51). Integrating with respect to u and retaining only leading order terms yields

$$(8.30) \qquad F_7 = \frac{5}{48} \pi^2 \frac{k_o^2}{k_n^4} \quad .$$

To evaluate F_1 we first eliminate the factor $\vec{k}_o \cdot \vec{p}$ by means of (8.16) and write

$$(8.31) \qquad F_1 = - 12k_o \frac{\partial}{\partial k_o} \int \frac{x_2 x_3 \delta(x_1 + x_2 + x_3 - 1) dx_1 dx_2 dx_3 d\tau_p}{(p^2 x_1 + A_1 x_2 + A_2 x_3)^5} \quad .$$

Integrating with respect to x_1 and $d\tau_p$ we obtain

$$(8.32) \qquad F_1 = \frac{15\pi^2}{16} \frac{\partial}{\partial k_o} \int \frac{x_2 x_3 dx_2 dx_3}{C^{7/2}} \quad .$$

Differentiation with respect to k_o yields

$$(8.33) \qquad F_1 = \frac{105\pi^2}{16} \int \frac{x_2^2 x_3 (k_o^2 - k_o^2 x_2 - \vec{k}_o \cdot \vec{k}_n x_3) dx_2 dx_3}{C^{9/2}} \quad .$$

Changing to u, v variables and noting that odd powers of v disappear upon integration with respect to v, we obtain

$$(8.34) \qquad F_1 = 105\pi^2 \frac{k_o^2}{q^9} \int_0^1 \int_{-u}^{u} \frac{\left[2(u^3 - uv^2) - (u^4 - v^4)\right]}{(\omega - v^2)^{9/2}} \, du \, dv$$

$$- 105\pi^2 \frac{\vec{k}_o \cdot \vec{k}_n}{q^9} \int_0^1 \int_{-u}^{u} \frac{(u^4 - 2u^2 v^2 + v^4)}{(\omega - v^2)^{9/2}} \, du \, dv \quad ,$$

where

$$(8.35)\qquad \omega = \frac{1}{q^2}\left[2(k_o^2 + k_n^2 + \tfrac{5}{4})u - (k_o^2 + k_n^2 + 2\vec{k}_o\cdot\vec{k}_n)u^2\right] .$$

For $\theta = 0$ the v^2 term in the denominator may be neglected, and $\vec{k}_o \cdot \vec{k}_n$ may be replaced by $k_o^2 - \frac{3}{8}$; then (8.34) becomes

$$(8.36)\qquad F_1 = 210\pi^2 k_o^2 \int_o^1\int_{-u}^{u} \frac{u(1-u)(u^2 - v^2)}{(\omega q^2)^{9/2}}\,du\,dv + \frac{315\pi^2}{8}\int_o^1\int_{-u}^{u} \frac{(u^4 - 2u^2v^2 + v^4)}{(\omega q^2)^{9/2}}\,du\,dv$$

Integration with respect to v yields

$$(8.37)\qquad F_1 = 280\pi^2 k_o^2 \int_o^1 \frac{(u^4 - u^5)}{(\omega q^2)^{9/2}}\,du + 42\pi^2 \int_o^1 \frac{u^5}{(\omega q^2)^{9/2}}\,du .$$

Because of a cancellation occurring in the leading order term of the first integral on the right-hand side of (8.37), the second integral may not be neglected. Carrying out the integration and retaining leading order terms we obtain

$$(8.38)\qquad F_1 = \frac{1}{10}\,\frac{\pi^2}{k_o^2}\left(\frac{8}{5}\right)^{5/2} .$$

To obtain the integral F_2 we first write F_2 in terms of x coordinates:

$$(8.39)\qquad F_2 = 24\int \frac{x_2x_3\,\delta(x_1 + x_2 + x_3 - 1)\,dx_1\,dx_2\,dx_3\,d\tau_p}{(p^2x_1 + A_1x_2 + A_2x_3)^5} .$$

Integration with respect to x_1 and $d\tau_p$ yields

$$(8.40)\qquad F_2 = \frac{15\pi^2}{8}\int \frac{x_2x_3\,dx_2\,dx_3}{C^{7/2}} .$$

Changing to u, v variables we obtain

$$(8.41) \qquad F_2 = \frac{30\pi^2}{q^7} \int_0^1 \int_{-u}^{u} \frac{(u^2 - v^2)}{(\omega - v^2)^{7/2}} \, dudv \ .$$

Neglecting the v^2 term in the denominator and integrating with respect to v, we have

$$(8.42) \qquad F_2 = 40\pi^2 \int_0^1 \frac{u^3}{(\omega q^2)^{7/2}} \, du \ .$$

Integrating with respect to u and retaining leading order terms gives finally

$$(8.43) \qquad F_2 = \frac{1}{8} \frac{\pi^2}{k_o^2} \left(\frac{8}{5}\right)^{5/2} .$$

Substitution of the values obtained for the integrals F_1, F_2, F_4, F_6, and F_7 in (8.25) for the scattering amplitude yields

$$(8.44) \qquad g_s' = \frac{\sqrt{2}\, i}{k_o^3} \left(\frac{85}{24} - \frac{16}{125} \left(\frac{8}{5}\right)^{1/2} \right) .$$

Here the second term on the right-hand side is small compared to the first term and the equation therefore may be approximated by

$$(8.45) \qquad g_s' = \frac{85}{24} \frac{\sqrt{2}\, i}{k_o^3} \ .$$

The result given by Corinaldesi and Trainor for g_a' evaluated for the limiting case $\theta = 0$ has the form

$$(8.46) \qquad g_a' = \frac{32}{27} \frac{\sqrt{2}\, i}{k_o^3} \ .$$

Equations (8.45) and (8.46) show that the scattered amplitudes obtained by the two perturbation schemes agree with regard to energy dependence for $\theta = 0$ in 2P exchange scattering. This is in contrast to the result for $\theta = 0$ in 2S exchange scattering where the terms furnished by the asymmetric scheme are negligible compared to those obtained with the symmetric method.

For convenience and reference the results obtained in the present investigation are summarized in Table I.

Table I

Type of Scattering	Symmetric Method			Asymmetric Method		
		$k^2\mu \gg 1$	$\mu = 0$		$k^2\mu \gg 1$	$\mu = 0$
2S direct	f_s	$\frac{8}{81} \frac{\sqrt{2}\, i}{k_o^3 \mu}$	$\frac{512}{729} \sqrt{2}$	f_a	$\frac{1}{8} \frac{\sqrt{2}}{k_o^6 \mu^3}$	$\frac{512}{729} \sqrt{2}$
2S exchange	g_s	$\frac{173}{64} \frac{\sqrt{2}\, i}{k_o^5 \mu}$	$\frac{47}{16} \frac{\sqrt{2}\, i}{k_o^3}$	g_a	$- \frac{\sqrt{2}}{2k_o^6} (8 - \frac{1}{\mu^2})$	$\frac{40 \sqrt{2}}{81} \frac{1}{k_o^4}$
2P direct	f_s'	$\frac{4}{27} \frac{\sqrt{2}\, i(1-2u)}{k_o^3 \mu}$	$\frac{1536}{729} ik_o$	f_a'	$\frac{3 \sqrt{2}}{32} \frac{i}{k_o^7 \mu^3}$	$\frac{1536}{729} ik_o$
2P exchange	g_s'	$- \frac{45}{32} \frac{\sqrt{2}\, i}{k_o^5 \mu}$	$\frac{85}{24} \frac{\sqrt{2}\, i}{k_o^3}$	g_a'	$- \frac{\sqrt{2}\, i}{8 k_o^7} (32 - \frac{5}{\mu^2})$	$\frac{32}{27} \frac{\sqrt{2}\, i}{k_o^3}$

9. Discussion

The results obtained in this investigation indicate substantial differencesbetween the two perturbation schemes for the case of inelastic scattering. Whereas for elastic scattering the results differ only for exchange scattering, for the inelastic case differences occur both for direct and exchange scattering.

One of the important results obtained with the symmetric perturbation method is that the exchange scattering is small compared to direct scattering, for inelastic as well as elastic processes. The asymmetric method yields direct and exchange scattering amplitudes of the same order of magnitude for inelastic scattering. The results of the present method appear more plausible since we would expect the effect of exchange to become less important as the energy of the incident electron increases.

Of considerable interest are the results for direct scattering at small scattering angles. The calculations by the symmetric method for this case agree with those by the asymmetric method. Since the total scattered cross-section for high-speed electrons depends principally upon small-angle scattering, total cross-sections obtained from asymmetric perturbation calculations may be substantially correct.

The difference in results obtained by the two different perturbation schemes may perhaps be understood more clearly if the Coulomb wave functions occurring in the symmetric method are expressed in their asymptotic form, as the sum of a plane wave and a spherical wave [4]. The product of the two Coulomb wave functions occurring in the symmetric perturbation integral then consists of three types of terms: (1) a plane-plane wave term, (2) a plane-spherical wave term and (3) a spherical-spherical wave term. In the asymmetric perturbation integral only term (1) occurs; terms (2) and (3) represent the effect of the spherical wave portion of the Coulomb wave function. An order of magnitude calculation indicates that term (3) is always negligible compared to term (2). For inelastic scattering term (2) is generally dominant over term (1), and the asymmetric method is in error, therefore, by omission of term (2). For direct elastic scattering, however, term (1) is dominant over term (2) so that for this case the asymmetric method yields accurate results.

The calculations with the symmetric perturbation are considerably more difficult than those with the asymmetric perturbation, particularly in the case of exchange scattering. The present investigation indicates, however, that exchange scattering may be negligible compared to direct scattering. In addition, the calculations for direct scattering may be considerably simplified without loss of essential accuracy, by the method of altered states. Calculations using the method of the symmetric perturbation should not, therefore, be forbiddingly greater than those using the method of the asymmetric perturbation.

As indicated by Borowitz[2] an experiment which would measure the elastic scattering cross section would not determine which perturbation scheme is the more accurate. For inelastic scattering, however, the substantial differences obtained for direct scattering indicate that a scattering experiment measuring the differential cross section for this process may enable us to determine which perturbation scheme is valid.

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